# ANALYSIS II EXAMPLES 3

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The questions on this sheet are not all equally difficult and the harder ones are marked with $*$'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`. The questions are based on the example sheets I gave last year, but I have made a few changes.

**1**. Consider the mapping $\Omega \colon \mathbb{R}^6 \to \mathbb{R}^3$ defined by $\Omega(\mathbf{x}, \mathbf{y}) = \mathbf{x} \wedge \mathbf{y}$ (i.e. the usual 'cross product' of three-dimensional vectors). Prove directly from the definition that $\Omega$ is differentiable everywhere, and express its derivative at $(\mathbf{x}, \mathbf{y})$ first as a linear map and then as a Jacobian matrix.

**2**. At which points of $\mathbb{R}^2$ are the following functions $\mathbb{R}^2 \to \mathbb{R}$ differentiable?
   (i) $f(x, y) = xy\,|x - y|$.
   (ii) $f(x, y) = xy/\sqrt{(x^2 + y^2)}$   $((x, y) \neq (0, 0))$, $f(0, 0) = 0$.
   (iii) $f(x, y) = xy \sin 1/x$   $(x \neq 0)$, $f(0, y) = 0$.

**3**. (i) Let $V$ be a finite dimensional real vector space equipped with an inner product $\langle -, - \rangle$, and let $\| - \|$ be the norm derived from this inner product (i.e. $\|x\| = \sqrt{\langle x, x \rangle}$). Show that the function $V \to \mathbb{R}$ sending $x$ to $\|x\|$ is differentiable at all nonzero $x \in V$. [Hint: first show that $x \mapsto \|x\|^2$ is differentiable.]
   (ii) At which points in $\mathbb{R}^2$ are the functions $\| - \|_1$ and $\| - \|_\infty$ differentiable? [The shapes of the unit balls give a clue to where differentiability can be expected to fail.]

**4**. Let $f(x, y) = x^2 y/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Show that $f$ is continuous at $(0, 0)$, and that it has directional derivatives in all directions there (i.e., for any fixed $\alpha$, the function $t \mapsto f(t \cos \alpha, t \sin \alpha)$ is differentiable at $t = 0$). Is $f$ differentiable at $(0, 0)$?

**5**. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $g(x) = f(x, c - x)$ where $c$ is a constant. Show that $g : \mathbb{R} \to \mathbb{R}$ is differentiable and find its derivative
   (i) directly from the definition of differentiability
and also
   (ii) by using the chain rule.
Deduce that if $D_1 f = D_2 f$ throughout $\mathbb{R}^2$ then $f(x, y) = h(x + y)$ for some differentiable function $h$.

**6**. We work in $\mathbb{R}^3$ with the usual inner product. Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x) = \frac{x}{\|x\|} \quad \text{for } x \neq 0$$

and $f(0) = 0$. Show that $f$ is differentiable except at $0$ and

$$Df(x)(h) = \frac{h}{\|x\|} - \langle x, h \rangle \frac{x}{\|x\|^3}.$$

Verify that $Df(x)(h)$ is orthogonal to $x$ and explain geometrically why this is the case.

**7**. Put $f(0, 0) = 0$, and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x, y) \neq (0, 0)$. Prove that

(a) $f$, $D_1 f$, $D_2 f$ are continuous in $\mathbb{R}^2$;
(b) $D_{12} f$ and $D_{21} f$ exist at every point in $\mathbb{R}^2$, and are continuous except at $(0,0)$;
(c) $D_{12} f(0,0) = 1$ and $D_{21} f(0,0) = -1$.

**8**. Let $E \subset \mathbb{R}^2$ be an open set that contains the rectangle $[a, b] \times [c, d]$. Suppose that $g : E \to \mathbb{R}$ is continuous and that the partial derivative $D_2 g$ exists and is continuous. Set

$$G(y) = \int_a^b g(x, y) \, dx.$$

Show that $G$ is differentiable on $(c, d)$ with derivative

$$G'(y) = \int_a^b D_2 g(x, y) \, dx.$$

**9**. [Tripos IB 98210, modified] Let $V$ be the space of linear maps $\mathbb{R}^n \to \mathbb{R}^n$, equipped with the operator norm (cf. questions 9 and 10 on sheet 2). Consider the function $f : V \to V$ defined by $f(\alpha) = \alpha^2$: show that $f$ is differentiable everywhere in $V$. Is it true that $f'(\alpha) = 2\alpha$? If not, what is the derivative of $f$ at $\alpha$?

Now let $U \subseteq V$ be the open subset consisting of invertible endomorphisms, and let $g : U \to V$ be defined by $g(\alpha) = \alpha^{-1}$. Show that $g$ is differentiable at $\iota$ (the identity mapping), and that its derivative at $\iota$ is the linear mapping $V \to V$ which sends $\beta$ to $-\beta$. Suppose now that $\alpha$ and $\alpha + \gamma$ are both in $U$; verify that

$$(\alpha + \gamma)^{-1} - \alpha^{-1} = [(\iota + \alpha^{-1}\gamma)^{-1} - \iota]\alpha^{-1} \ .$$

Hence, or otherwise, show that $g$ is differentiable at $\alpha$, and find its derivative there.

**\*10**. Let $M_n(\mathbb{R})$ denote the vector space of all $(n \times n)$ real matrices, equipped with any suitable norm. By considering $\det(I + H)$ as a polynomial in the entries of $H$, show that the function $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is differentiable at the identity matrix $I$ and that its derivative there is the function $H \mapsto \operatorname{tr} H$. Hence show that $\det$ is differentiable at any invertible matrix $A$, with derivative $H \mapsto \det A \operatorname{tr}(A^{-1}H)$. Recalling from question 10 on sheet 2 that all matrices sufficiently close to the identity matrix are invertible, calculate the second derivative of $\det$ at $I$ as a bilinear map $M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}$, and verify that it is symmetric.

**11**. If $f$ is a real function defined in a convex open set $E \subset \mathbb{R}^n$ such that $D_1 f(x) = 0$ for all $x \in E$, prove that $f(x)$ only depends on $x_2, \ldots, x_n$. What can you say if $E$ is not convex? [Recall that $E$ is said to be convex if $\lambda x + (1 - \lambda)y \in E$ whenever $x \in E$, $y \in E$ and $\lambda \in (0, 1)$.]