ANALYSIS II EXAMPLES 3

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The questions on this sheet are not all equally difficult and the harder ones are marked with $*$'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`. The questions are based on the example sheets I gave last year, but I have made a few changes.

1. Consider the mapping $\Omega\colon \mathbb{R}^6 \to \mathbb{R}^3$ defined by $\Omega(\mathbf{x}, \mathbf{y}) = \mathbf{x} \wedge \mathbf{y}$ (i.e. the usual 'cross product' of three-dimensional vectors). Prove directly from the definition that Ω is differentiable everywhere, and express its derivative at $(\mathbf{x}, \mathbf{y})$ first as a linear map and then as a Jacobian matrix.

2. At which points of $\mathbb{R}^2$ are the following functions $\mathbb{R}^2 \to \mathbb{R}$ differentiable?
(i) $f(x, y) = xy\,|x - y|$.
(ii) $f(x, y) = xy/\sqrt{(x^2 + y^2)} \quad ((x, y) \neq (0, 0))$, $f(0, 0) = 0$.
(iii) $f(x, y) = xy \sin 1/x \quad (x \neq 0)$, $f(0, y) = 0$.

3. (i) Let V be a finite dimensional real vector space equipped with an inner product $\langle -, - \rangle$, and let $\| - \|$ be the norm derived from this inner product (i.e. $\|x\| = \sqrt{\langle x, x\rangle}$). Show that the function $V \to \mathbb{R}$ sending x to $\|x\|$ is differentiable at all nonzero $x \in V$. [Hint: first show that $x \mapsto \|x\|^2$ is differentiable.]
(ii) At which points in $\mathbb{R}^2$ are the functions $\| - \|_1$ and $\| - \|_\infty$ differentiable? [The shapes of the unit balls give a clue to where differentiability can be expected to fail.]

4. Let $f(x, y) = x^2y/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Show that f is continuous at $(0, 0)$, and that it has directional derivatives in all directions there (i.e., for any fixed α, the function $t \mapsto f(t \cos \alpha, t \sin \alpha)$ is differentiable at $t = 0$). Is f differentiable at $(0, 0)$?

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $g(x) = f(x, c - x)$ where c is a constant. Show that $g : \mathbb{R} \to \mathbb{R}$ is differentiable and find its derivative
(i) directly from the definition of differentiability
and also
(ii) by using the chain rule.
Deduce that if $D_1 f = D_2 f$ throughout $\mathbb{R}^2$ then $f(x, y) = h(x + y)$ for some differentiable function h.

6. We work in $\mathbb{R}^3$ with the usual inner product. Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x) = \frac{x}{\|x\|} \quad \text{for } x \neq 0$$

and $f(0) = 0$. Show that f is differentiable except at 0 and

$$Df(x)(h) = \frac{h}{\|x\|} - \langle x, h\rangle \frac{x}{\|x\|^3}.$$

Verify that $Df(x)(h)$ is orthogonal to h and explain geometrically why this is the case.

7. Put $f(0, 0) = 0$, and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x, y) \neq (0, 0)$. Prove that

(a) f, $D_1 f$, $D_2 f$ are continuous in $\mathbb{R}^2$;
(b) $D_{12} f$ and $D_{21} f$ exist at every point in $\mathbb{R}^2$, and are continuous except at $(0,0)$;
(c) $D_{12} f(0,0) = 1$ and $D_{21} f(0,0) = -1$.

8. [Tripos IB 98210, modified] Let V be the space of linear maps $\mathbb{R}^n \to \mathbb{R}^n$, equipped with the operator norm (cf. questions 9 and 10 on sheet 2). Consider the function $f : V \to V$ defined by $f(\alpha) = \alpha^2$: show that f is differentiable everywhere in V. Is it true that $f'(\alpha) = 2\alpha$? If not, what is the derivative of f at α?

Now let $U \subseteq V$ be the open subset consisting of invertible endomorphisms, and let $g : U \to V$ be defined by $g(\alpha) = \alpha^{-1}$. Show that g is differentiable at ι (the identity mapping), and that its derivative at ι is the linear mapping $V \to V$ which sends β to $-\beta$. Suppose now that α and $\alpha + \gamma$ are both in U; verify that

$$(\alpha + \gamma)^{-1} - \alpha^{-1} = [(\iota + \alpha^{-1}\gamma)^{-1} - \iota]\alpha^{-1} \ .$$

Hence, or otherwise, show that g is differentiable at α, and find its derivative there.

***9**. Let $M_n(\mathbb{R})$ denote the vector space of all $(n \times n)$ real matrices, equipped with any suitable norm. By considering $\det(I + H)$ as a polynomial in the entries of H, show that the function $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is differentiable at the identity matrix I and that its derivative there is the function $H \mapsto \operatorname{tr} H$. Hence show that det is differentiable at any invertible matrix A, with derivative $H \mapsto \det A \operatorname{tr}(A^{-1}H)$. Recalling from question 10 on sheet 2 that all matrices sufficiently close to the identity matrix are invertible, calculate the second derivative of det at I as a bilinear map $M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}$, and verify that it is symmetric.

10. If f is a real function defined in a convex open set $E \subset \mathbb{R}^n$ such that $D_1 f(x) = 0$ for all $x \in E$, prove that $f(x)$ only depends on $x_2, \ldots, x_n$. What can you say if E is not convex? [Recall that E is said to be convex if $\lambda x + (1 - \lambda) y \in E$ whenever $x \in E$, $y \in E$ and $\lambda \in (0,1)$.]