ANALYSIS II EXAMPLES 2

G.P. Paternain Mich. 2003

The questions on this sheet are not all equally difficult and the harder ones are marked with $*$'s. In all the questions on this sheet, the norm $\| - \|$ on $\mathbb{R}^n$ may be taken to be whichever of the three norms $\| - \|_1$, $\| - \|_2$ or $\| - \|_\infty$ you find most convenient to work with. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Prove the following facts about convergence of sequences in an arbitrary normed space V:
(i) If $(x_n) \to x$ and $(y_n) \to y$, then $(x_n + y_n) \to x + y$.
(ii) If $(x_n) \to x$ and $\lambda \in \mathbb{R}$, then $(\lambda x_n) \to \lambda x$.
(iii) If $x_n = x$ for all $n \geq n_0$, then $(x_n) \to x$.
(iv) If $(x_n) \to x$, then any subsequence (x_{n_i}) of (x_n) also converges to x.

2. Which of the following subsets of $\mathbb{R}^2$ are (a) open, (b) closed?
(i) $\{(x, 0) : 0 \leq x \leq 1\}$.
(ii) $\{(x, 0) : 0 < x < 1\}$.
(iii) $\{(x, y) : y \neq 0\}$.
(iv) $\{(x, y) : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$.
(v) $\{(x, y) : xy = 1\}$.

***3**. Let E be a subset of $\mathbb{R}^n$ (or, if you prefer, of an arbitrary normed space). We define the *closure* $\overline{E}$ of E to be the set of all points which can occur as limits of sequences of points of E, and the *interior* E° of E to be the set

$$\{x \in \mathbb{R}^n : (\exists \epsilon > 0)(B(x, \epsilon) \subseteq E)\} .$$

(i) Show that $\overline{E}$ is closed, and that it is the smallest closed set containing E.
(ii) Show that E° is open, and that it is the largest open set contained in E.
(iii) Show that $\overline{\mathbb{R}^n \setminus E} = \mathbb{R}^n \setminus E^\circ$.
(iv) By considering the inclusion relations which must hold amongst the sets

$$\ldots, \overline{(\overline{E})^\circ}, (\overline{E})^\circ, \overline{E}, E, E^\circ, \overline{E^\circ}, \ldots$$

show that starting from a given E, it is not possible to produce more than seven distinct sets by repeated application of the operators $\overline{(-)}$ and $(-)^\circ$.
(v) Find an example of a set in $\mathbb{R}^1$ which does give rise to seven distinct sets in this way.

4. Let E be a subset of $\mathbb{R}^n$ which is both open and closed. Show that E is either the whole of $\mathbb{R}^n$ or the empty set. [Method: suppose for a contradiction that $x \in E$ but $y \in \mathbb{R}^n \setminus E$. Define a function $f : [0, 1] \to \mathbb{R}$ by setting $f(t) = 1$ if the point $tx + (1 - t)y$ belongs to E, and $f(t) = 0$ otherwise; now recall a suitable theorem from Analysis I.]

5. (i) Show that the mapping $\mathbb{R}^{2n} \to \mathbb{R}^n$ which sends a $2n$-dimensional vector

$$(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$$

to

$$(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

is continuous. Deduce that if f and g are continuous functions from (a subset of) $\mathbb{R}^p$ to $\mathbb{R}^n$, so is their (pointwise) sum $f + g$.

(ii) By considering a suitable function $\mathbb{R}^{n+1} \to \mathbb{R}^n$, give a similar proof that if f is a continuous $\mathbb{R}^n$-valued function on a subset E of $\mathbb{R}^p$, and λ is a continuous real-valued function on E, then the pointwise scalar product λf (i.e. the function whose value at x is $\lambda(x).f(x)$) is continuous on E.

6. If A and B are subsets of $\mathbb{R}^n$, we write $A + B$ for the set $\{a + b : a \in A, b \in B\}$. Show that if A and B are both closed and one of them is bounded, then $A + B$ is closed. Give an example in $\mathbb{R}^1$ to show that the boundedness condition cannot be omitted. If A and B are both open, is $A + B$ necessarily open? Justify your answer.

7. Let $f : \mathbb{R}^n \to \mathbb{R}^p$, and let E, F be subsets of $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively. Determine which of the following statements are always true and which may be false, giving a proof or a counterexample as appropriate. [N.B.: for the counterexamples, it suffices to take $n = p = 1$.]

(i) If $f^{-1}(F)$ is closed whenever F is closed, then f is continuous.
(ii) If f is continuous, then $f^{-1}(F)$ is closed whenever F is closed.
(iii) If f is continuous, then $f(E)$ is open whenever E is open.
(iv) If f is continuous, then $f(E)$ is bounded whenever E is bounded.
(v) If $f(E)$ is bounded whenever E is bounded, then f is continuous.

8. In lectures we proved that if E is a closed and bounded set in $\mathbb{R}^n$, then any continuous function defined on E has bounded image. Prove the converse: if every continuous real-valued function on $E \subseteq \mathbb{R}^n$ is bounded, then E is closed and bounded.

9. Let $\theta \colon \mathbb{R}^n \to \mathbb{R}^p$ be a linear map. Show that

$$\sup\{\|\theta(x)\| : x \in \mathbb{R}^n, \|x\| \le 1\} = \inf\{k \in \mathbb{R} : k \text{ is a Lipschitz constant for } \theta\} .$$

Show also that the function which sends θ to the common value of these two expressions is a norm on the vector space $V = \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ of all linear maps $\mathbb{R}^n \to \mathbb{R}^p$. [We call this function the *operator norm* on V.]

10. Let V be the vector space of all linear maps $\mathbb{R}^n \to \mathbb{R}^p$, equipped with the operator norm defined in the previous question.

(i) Show that if $\|\theta\| < \epsilon$ then all the entries in the matrix representing θ (with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^p$) have absolute value less than ϵ.

(ii) Conversely, if all entries of the matrix A have absolute value less than ϵ, show that the norm of the linear map represented by A is less than $np\epsilon$. Deduce that convergence for sequences of linear maps is equivalent to 'entry-wise' convergence of the representing matrices, and in particular that V is complete.

(iii) If θ and ϕ are two composable linear maps, show that the norm of the composite $\theta \circ \phi$ is less than or equal to the product $\|\theta\|.\|\phi\|$.

(iv) Now specialize to the case $n = p$. Show that if θ is an endomorphism of $\mathbb{R}^n$ satisfying $\|\theta\| < 1$, then the sequence whose mth term is $\iota + \theta + \theta^2 + \cdots + \theta^{m-1}$ converges (here ι denotes the identity mapping), and deduce that $\iota - \theta$ is invertible.

(v) Deduce that if α is invertible then so is $\alpha - \beta$ whenever $\|\beta\| < \|\alpha^{-1}\|^{-1}$, and hence that the set of invertible linear maps is open in V.

11. Let ℓ_0 be the space of all real sequences $(a_n)_{n=1}^{\infty}$ such that all but finitely many of the a_n are zero. If we use the natural definitions of addition and scalar mutliplication

$$(a_n) + (b_n) = (a_n + b_n), \quad \lambda(a_n) = (\lambda\, a_n)$$

then ℓ_0 is a vector space. Find two norms in ℓ_0 which are not Lipschitz equivalent. Can you find uncountably many which are not Lipschitz equivalent?