ANALYSIS II—EXAMPLES 4 Mich. 2018

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1. Quickies: (a) Let $F : [0,1] \times \mathbb{R}^m \to \mathbb{R}$ be continuous and $a = (a_0, \dots, a_{m-1}) \in \mathbb{R}^m$. Suppose that F is uniformly Lipschitz in the $\mathbb{R}^m$ variables near a, i.e. for some constant K and an open subset U of $\mathbb{R}^m$ containing a, $|F(t,x) - F(t,y)| \leq K\|x-y\|$ for all $t \in [0,1]$, $x, y \in U$. Use the Picard–Lindelöf existence theorem for first order ODE systems to show that there is an $\epsilon > 0$ such that, writing $f^{(j)}$ for the jth derivative of f, the mth order initial value problem

$$f^{(m)}(t) = F(t, f(t), f^{(1)}(t), \dots, f^{(m-1)}(t)) \quad \text{for} \quad t \in [0,\epsilon);$$

$$f^{(j)}(0) = a_j \quad \text{for} \quad 0 \leq j \leq m-1$$

has a unique C^m solution $f : [0,\epsilon) \to \mathbb{R}$ (see also Q2 below).

(b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $a \in \mathbb{R}^n$. If the directional derivatives $D_u f(a)$ exist for all directions $u \in \mathbb{R}^2$ and if $D_u f(a)$ depends linearly on u, does it follow that f is differentiable at a?

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If f is differentiable at $0 \in \mathbb{R}^2$, and if the partial derivatives of f exist in a neighbourhood of 0, does it follow that one partial derivative is continuous at 0?

(d) Let $f\colon [a,b] \to \mathbb{R}^2$ be continuous, and differentiable on (a,b). Does it follow that there exists $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$?

2. Let $F : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous, $x_0 \in \mathbb{R}^n$ and $R > 0$. Suppose that $\sup_{[a,b]\times\overline{B_R(x_0)}} \|F\| \leq R(b-a)^{-1}$ and that $\|F(t,x) - F(t,y)\| \leq K\|x-y\|$ for some K and all $t \in [a,b]$, $x, y \in \overline{B_R(x_0)}$. We showed in lecture that for each $t_0 \in [a,b]$, there is a unique $f \in C([a,b]; \overline{B_R(x_0)})$ solving the integral equation $f(t) = x_0 + \int_{t_0}^t F(s, f(s))\, ds$, $t \in [a,b]$. Show that this f is in fact the unique function in $C([a,b];\mathbb{R}^n)$ solving the integral equation. (Hint: for $g \in C([a,b];\mathbb{R}^n)$ solving $g(t) = x_0 + \int_{t_0}^t F(s, g(s))\, ds$, $t \in [a,b]$, let $\Lambda^+ = \{t \in [t_0, b] : \|g(\sigma) - x_0\| \leq R \ \ \forall \sigma \in [t_0, t]\}$ and consider the possibility that $\sup \Lambda^+ < b$.)

3. Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $a \in U$. A differentiable curve passing through a is a differentiable map $\gamma : (-1,1) \to \mathbb{R}^n$ with $\gamma(0) = a$. If $f \circ \gamma$ is differentiable at 0 for every differentiable curve γ passing through a, does it follow that f is differentiable at a?

4. Define $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x,y,z) = (e^{x+y+z}, \cos x^2 y)$. Without making use of partial derivatives, show that f is everywhere differentiable and find $Df(a)$ at each $a \in \mathbb{R}^3$. Find all partial derivatives of f and hence, using appropriate results on partial derivatives, give an alternative proof of this result.

5. Consider the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ for $x \neq 0$, and $f(0) = 0$. Show that f is differentiable except at 0, and that

$$Df(x)(h) = \frac{h}{\|x\|} - \frac{x(x \cdot h)}{\|x\|^3}.$$

Verify that $Df(x)(h)$ is orthogonal to x and explain geometrically why this is the case.

6. At which points of $\mathbb{R}^2$ is the function $f(x,y) = |x||y|$ differentiable? What about the function $g: \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x,y) = xy/\sqrt{x^2+y^2}$ if $(x,y) \neq (0,0)$, $g(0,0) = 0$?

7. Let f be a real-valued function on an open subset U of $\mathbb{R}^2$ such that that $f(\cdot, y)$ is continuous for each fixed $y \in U$ and $f(x, \cdot)$ is continuous for each fixed $x \in U$. Give an example to show that f need not be continuous on U. If additionally $f(\cdot, y)$ is Lipschitz for each $y \in U$ with Lipschitz constant independent of y, show that f is continuous on U. Deduce that if $D_1 f$ exists and is bounded on U and $f(x, \cdot)$ is continuous for each fixed $x \in U$, then f is continuous on U.

8. Let $f: \mathbb{R}^2 \to \mathbb{R}$ and $a \in \mathbb{R}^2$. If $D_1 f$ exists in some open ball around a and is continuous at a, and if $D_2 f$ exists at a, show that f is differentiable at a.

9. (*Some useful properties of the operator norm*). Recall that the operator norm on the vector space $\mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ is defined by $\|A\|_{op} = \sup_{x \in S} \|A(x)\|$ where $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Prove the following: (i) if $A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ then $\|A\|_{op} = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|A(x)\|}{\|x\|}$; (ii) if $A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ and $B \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^p)$ then $B \circ A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^p)$ and $\|B \circ A\|_{op} \leq \|B\|_{op} \|A\|_{op}$; (iii) if $A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R})$ then there is $a \in \mathbb{R}^n$ such that $Ax =< a, x >$ for all $x \in \mathbb{R}^n$ and in this case $\|A\|_{op} = \|a\|$; (iv) if $A \in \mathcal{L}(\mathbb{R}; \mathbb{R}^m)$ then there is $a \in \mathbb{R}^m$ such that $Ax = xa$ for all $x \in \mathbb{R}$ and in this case $\|A\|_{op} = \|a\|$; (v) if $A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ and (A_{ij}) is the matrix of A relative to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, then $\frac{1}{\sqrt{n}} \left(\sum_{j=1}^n \sum_{i=1}^m A_{ij}^2 \right)^{1/2} \leq \|A\|_{op} \leq \left(\sum_{j=1}^n \sum_{i=1}^m A_{ij}^2 \right)^{1/2}$, with equality in the right hand side inequality if and only if either $A = 0$ or $\operatorname{rank}(A) = 1$.

10. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Suppose that $\|Df(x) - I\| \leq \mu$ for some $\mu \in (0,1)$ and all $x \in \mathbb{R}^n$, where I is the identity map on $\mathbb{R}^n$ and $\|\cdot\|$ is the operator norm. Show that f is an open mapping, i.e. that f maps open subsets to open subsets. Show that $\|x - y\| \leq (1-\mu)^{-1} \|f(x) - f(y)\|$ for all $x, y \in \mathbb{R}^n$, and deduce that f is one-to-one and that $f(\mathbb{R}^n)$ is closed in $\mathbb{R}^n$. Conclude that f is a diffeomorphism of $\mathbb{R}^n$, i.e. that f is a bijection with C^1 inverse. What can you say about a C^1 map $f : \mathbb{R}^n \to \mathbb{R}^n$ assumed to satisfy only that $\|Df(x) - I\| < 1$ for all $x \in \mathbb{R}^n$?

11. Let $\mathcal{M}_n$ be the space of $n \times n$ real matrices equipped with a norm. Show that the determinant function $\det: \mathcal{M}_n \to \mathbb{R}$ is differentiable at the identity matrix I with $D\det(I)(H) = \operatorname{tr}(H)$. Deduce that det is differentiable at any invertible matrix A with $D\det(A)(H) = \det A \operatorname{tr}(A^{-1}H)$. Show further that det is twice differentiable at I and find $D^2 \det(I)$ as a bilinear map.

12. Define $f: \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^2$. Show that f is continuously differentiable on the whole of $\mathcal{M}_n$. Deduce that there is a continuous square-root function on some neighbourhood of I; that is, show that there is an open ball $B_\varepsilon(I)$ for some $\varepsilon > 0$ and a

continuous function $g\colon B_\varepsilon(I) \to \mathcal{M}_n$ such that $g(A)^2 = A$ for all $A \in B_\varepsilon(I)$. Is it possible to define a continuous square-root function on the whole of $\mathcal{M}_n$?

13. Let $C = \{(x,y) \in \mathbb{R}^2 : x^3 + y^3 - 3xy = 0\}$. Define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $F(x,y) = (x, x^3 + y^3 - 3xy)$. Show that F is locally C^1-invertible around each point of $C \setminus \{(0,0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$; that is, show that if $(x_0, y_0) \in C \setminus \{(0,0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$ then there are open sets U containing (x_0, y_0) and V containing $F(x_0, y_0) = (x_0, 0)$ such that F maps U bijectively to V with inverse a C^1 function. What is the derivative of the inverse function? Deduce that for each point $(x_0, y_0) \in C \setminus \{(0,0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$, there exists an open subset $I \subset \mathbb{R}$ containing x_0 and a C^1 function $g\colon I \to \mathbb{R}$ such that $C \cap U = \operatorname{graph} g \equiv \{(x, g(x)) : x \in I\}$.

14*. (i) Let f be a real-valued C^2 function on an open subset U of $\mathbb{R}^2$. If f has a local maximum at a point $a \in U$ (meaning that there is $\rho > 0$ such that $B_\rho(a) \subset U$ and $f(x) \le f(a)$ for every $x \in B_\rho(a)$), show that $Df(a) = 0$ and that the matrix $H = (D_{ij}f(a))$ is negative semi-definite (i.e. has non-positive eigenvalues).

(ii) Let U be a bounded open subset of $\mathbb{R}^2$ and let $f\colon \overline{U} \to \mathbb{R}$ be continuous on $\overline{U}$ (the closure of U) and C^2 in U. If f satisfies the partial differential inequality $\Delta f + aD_1 f + bD_2 f + cf \ge 0$ in U where Δ is the Laplace's operator defined by $\Delta f = D_{11}f + D_{22}f$, and a, b, c are real-valued functions on U with $c < 0$ on U, and if f is positive somewhere in $\overline{U}$, show that

$$\sup_{\overline{U}} f = \sup_{\partial U} f$$

where $\partial U = \overline{U} \setminus U$ is the boundary of U. Deduce that if a, b, c are as above, $\varphi\colon \partial U \to \mathbb{R}$ is a given continuous function, then for any $g\colon \mathbb{R}^2 \to \mathbb{R}$ there is at most one continuous function f on $\overline{U}$ that is C^2 in U and solves the boundary value problem $\Delta f + aD_1 f + bD_2 f + cf = g$ in U, $f = \varphi$ on ∂U.