ANALYSIS II—EXAMPLES 3 Mich. 2018

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1. Quickies: (a) Use the equivalence of norms on a finite dimensional vector space to show that for each n, there is a constant C such that the following holds: for every polynomial p of degree $\leq n$ there is $x_0 \in [0, 1/n]$ such that $|p(x)| \leq C|p(x_0)|$ for every $x \in [0, 1]$.

(b) If (X, d) is a metric space and A is a non-empty subset of X, show that the distance from $x \in X$ to A defined by $\rho(x) = \inf_{y \in A} d(x, y)$ is a Lipschitz function on X with Lipschitz constant ≤ 1.

(c) If (x_n), (y_n) are Cauchy sequences in a metric space (X, d), show that $(d(x_n, y_n))$ is convergent (in $\mathbb{R}$).

2. (a) Is the set $(1, 2]$ an open subset of the metric space $\mathbb{R}$ with the metric $d(x, y) = |x - y|$? Is it closed? What if we replace the metric space $\mathbb{R}$ with the space $[0, 2]$, the space$(1, 3)$ or the space $(1, 2]$, in each case with the metric d?

(b) Let X be a set equipped with the discrete metric, and Y any metric space. Describe all open subsets of X, closed subsets of X, compact subsets of X, Cauchy sequences in X, continuous functions $f : X \to Y$ and continuous functions $f : Y \to X$.

3. For each of the following sets X, determine whether the given function d defines a metric on X. In each case where the function does define a metric, describe the open ball $B_\varepsilon(x)$ for $x \in X$ and $\varepsilon > 0$ small.

(i) $X = \mathbb{R}^n$; $d(x, y) = \min\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$.

(ii) $X = \mathbb{Z}$; $d(x, x) = 0$, and, for $x \neq y$, $d(x, y) = 2^n$ where $x - y = 2^n a$ with n a non-negative integer and a an odd integer.

(iii) X is the set of functions from $\mathbb{N}$ to $\mathbb{N}$; $d(f, f) = 0$, and, for $f \neq g$, $d(f, g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

(iv) $X = \mathbb{C}$; $d(z, w) = |z - w|$ if z and w lie on the same line through the origin, $d(z, w) = |z| + |w|$ otherwise.

4. Let (X, d) be a metric space.

(a) Show that the union of any collection of open subsets of X must be open (regardless of whether the collection is finite, countable or uncountable), and that the intersection of any finite collection of open subsets is again open. Formulate and prove similar properties about the closed subsets of X.

(b) Let E be a subset of X. Show that there is a unique largest open subset E^o of X contained in E, i.e. a unique open subset E^o of X such that that $E^o \subseteq E$ and if G is any open subset of X with $G \subseteq E$ then $G \subseteq E^o$. The set E^o is called the *interior* of E in X. Show also that there is a unique smallest closed subset $\overline{E}$ of X containing E, i.e. a unique closed subset $\overline{E}$ of X with $E \subseteq \overline{E}$ and if F is any closed subset of X with $E \subseteq F$ then $\overline{E} \subseteq F$. The set $\overline{E}$ is called the *closure* of E in X.

(c) Show that

$$E^o = \{x \in X \,:\, B_\epsilon(x) \subset E \text{ for some } \epsilon > 0\}$$

and that

$$\overline{E} = \{x \in X \,:\, x_n \to x \text{ for some sequence } (x_n) \text{ in } E\}.$$

5. Let V be a normed space, $x \in V$ and $r > 0$. Prove that the closure of the open ball $B_r(x)$ is the closed ball $D_r(x) = \{y \in V : \|x - y\| \leq r\}$. Give an example to show that, in a general metric space (X, d), the closure of the open ball $B_r(x)$ need not be the closed ball $D_r(x) = \{y \in X : d(x, y) \leq r\}$.

6. In lectures we proved that if E is a compact subset of $\mathbb{R}^n$ with the Euclidean metric, then any continuous function on E has bounded image. Prove the converse: if E is a subset of $\mathbb{R}^n$ with the Euclidean metric and if every continuous function $f : E \to \mathbb{R}$ has bounded image, then E is compact.

7. Each of the following properties/notions makes sense for an arbitrary metric spaces X. Which are topological (i.e. dependent only on the collection of open subsets of X and not on the metric generating the open subsets)? Justify your answers.

(i) boundedness of a subset of X.

(ii) closed-ness of a subset of X.

(iii) notion that a subset of X is closed *and* bounded.

(iv) total boundedness of X; that is, the property that for every $\epsilon > 0$, there is a finite set $F \subset X$ such that the union of open balls with centres in F and radius ϵ is X.

(v) completeness of X.

(vi) notion that X is complete *and* totally bounded.

8. Use the Contraction Mapping Theorem to show that the equation $\cos x = x$ has a unique real solution. Find this solution to some reasonable accuracy using a calculator (remember to work in radians!), and justify the claimed accuracy of your approximation.

9. Let $I = [0, R]$ be an interval and let $C(I)$ be the space of continuous functions on I. Show that, for any $\alpha \in \mathbb{R}$, we may define a norm by $\|f\|_\alpha = \sup_{x\in I} |f(x)e^{-\alpha x}|$, and that the norm $\|\cdot\|_\alpha$ is Lipschitz equivalent to the uniform norm $\|f\| = \sup_{x\in I} |f(x)|$.
Now suppose that $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, and Lipschitz in the second variable. Consider the map T from $C(I)$ to itself sending f to $y_0 + \int_0^x \phi(t, f(t))dt$. Give an example to show that T need not be a contraction under the uniform norm. Show, however, that T is a contraction under the norm $\|\cdot\|_\alpha$ for some α, and hence deduce that the differential equation $f'(x) = \phi(x, f(x))$ has a unique solution on I satisfying $f(0) = y_0$.

10. Let (X, d) be a non-empty complete metric space. Suppose $f\colon X \to X$ is a contraction and $g\colon X \to X$ is a function which commutes with f, i.e. such that $f(g(x)) = g(f(x))$ for all $x \in X$. Show that g has a fixed point. Must this fixed point be unique?

11. Give an example of a non-empty complete metric space (X, d) and a function $f\colon X \to X$ satisfying $d(f(x), f(y)) < d(x, y)$ for all x, $y \in X$ with $x \neq y$, but such that f has no fixed point. Suppose now that X is a non-empty compact subset of $\mathbb{R}^n$ with the Euclidean metric. Show that in this case f must have a fixed point. If $g\colon X \to X$ satisfies $d(g(x), g(y)) \le d(x, y)$ for all x, $y \in X$, must g have a fixed point?

12. (a) Let $B = \overline{B_1(0)}$ be the closed unit ball in $\mathbb{R}^n$ and let $F : [0, 1] \times B \to \mathbb{R}^n$ be continuous. Suppose that there is a constant K such that $\|F(t, x) - F(t, y)\| \le K\|x - y\|$ for all $t \in [0, 1]$ and all $x, y \in B$. Let $x_1, x_2 \in \mathbb{R}^n$. By the Picard–Lindelöf theorem, we know that there is $\epsilon \in (0, 1]$ and differentiable functions $f_1, f_2 : [0, \epsilon] \to B$ such that $\frac{df_j}{dt} = F(t, f_j(t))$, $f_j(0) = x_j$ for $j = 1, 2$. Show that $\|f_1(t)-f_2(t)\| \le \|x_1-x_2\|e^{Kt}$ for all $t \in [0, \epsilon]$. (Notice that this in particular gives uniqueness of f satisfying $\frac{df}{dt} = F(t, f(t))$, $f(0) = x_0$ in some interval $[0, \epsilon]$, although such uniqueness is also automatically guaranteed by the Contraction Mapping Theorem we used to prove the existence of solutions).
(b) Now relax the above Lipschitz condition on F in the second variable to Hölder continuity, i.e. assume that there exist constants K and $\alpha \in (0, 1)$ such that $\|F(t, x)-F(t, y)\| \le K\|x-y\|^\alpha$ for all $t \in [0, 1]$ and all $x, y \in B$. If $f_1, f_2\, [0, \epsilon] \to \mathbb{R}^n$ are as above, show that $\|f_1(t) - f_2(t)\| \le \left(\|x_1 - x_2\|^{1-\alpha} + (1 - \alpha)Kt\right)^{\frac{1}{1-\alpha}}$ for all $t \in [0, \epsilon]$. What does this say about the set of solutions f to $\frac{df}{dt} = F(t, f(t))$, $f(0) = x_0$ in some interval $[0, \epsilon]$?

13.* Let (X, d) be a non-empty complete metric space and let $f\colon X \to X$ be a function such that for each positive integer n we have
(i) if $d(x, y) < n + 1$ then $d(f(x), f(y)) < n$; and
(ii) if $d(x, y) < 1/n$ then $d(f(x), f(y)) < 1/(n + 1)$.
Must f have a fixed point?

14.* Let K be a compact subset of $\mathbb{R}$ and let $p \in K$. Construct a metric d on $K_1 = K \setminus \{p\}$ such that (K_1, d) is complete and the topology generated by d on K_1 is the same as the topology generated by the Euclidean metric on K_1.

15.* (a) Let (X, d) be a totally bounded metric space. Show that any sequence (x_k) in X has a Cauchy subsequence. (b) Show that a metric space is compact if and only if it is complete and totally bounded.