

Please email comments, corrections to: n.wickramasekera@dpmmms.cam.ac.uk

1. Quickies: (a) If (f_n) is a sequence of real functions on the interval $[0, 1]$ converging uniformly to a function f on $[0, 1]$, and if f_n is continuous at $x_n \in [0, 1]$ with $x_n \rightarrow x$, does it follow that f is continuous at x ?
 (b) If (f_n) is a sequence of continuous real functions on the interval $[-1, 1]$ converging pointwise to a continuous function f on $[-1, 1]$, and if the convergence is uniform on $[-r, r]$ for every $r \in (0, 1)$, does it follow that the convergence is uniform on $[-1, 1]$?
 (c) If (f_n) is a sequence of real functions on the interval $[0, 1]$ converging uniformly to a function f on $[0, 1]$, and if each f_n is continuous except at countably many points, does it follow that there exists a point at which f is continuous?
 (d)* If (f_n) is a sequence of differentiable functions on $[0, 1]$ converging uniformly to a function f on $[0, 1]$, does it follow that there exists a point at which f is differentiable?
2. Which of the following sequences (f_n) of functions converge uniformly on the set X ?
 (a) $f_n(x) = x^n$ on $X = (0, 1)$; (b) $f_n(x) = x^n$ on $X = (0, \frac{1}{2})$; (c) $f_n(x) = xe^{-nx}$ on $X = [0, \infty)$;
 (d) $f_n(x) = e^{-x^2} \sin(x/n)$ on $X = \mathbb{R}$.
3. Let (f_n) and (g_n) be sequences of real-valued functions on a set E converging uniformly to f and g respectively. Show that the sequence of pointwise sums $(f_n + g_n)$ converges uniformly to $f + g$. On the other hand, show that the sequence of pointwise products $(f_n g_n)$ need not converge uniformly to fg , but if both f and g are bounded then $(f_n g_n)$ does converge uniformly to fg . What if f is bounded but g is not?
4. Let (f_n) be a sequence of bounded, real-valued functions on a set E converging uniformly to a function f . Show that f must be bounded. Give an example of a sequence (g_n) of bounded, real-valued functions on $[-1, 1]$ converging pointwise to a function g which is not bounded.
5. Let (f_n) be a sequence of real-valued continuous functions on a closed, bounded interval $[a, b]$, and suppose that f_n converges pointwise to a continuous function f . Show that if $f_n \rightarrow f$ uniformly and (x_n) is a sequence of points in $[a, b]$ with $x_n \rightarrow x$ then $f_n(x_n) \rightarrow f(x)$. On the other hand, show that if f_n does not converge uniformly to f then we can find a convergent sequence $x_n \rightarrow x$ in $[a, b]$ such that $f_n(x_n) \not\rightarrow f(x)$.
6. Let (f_n) be a sequence of real-valued functions on $[0, 1]$ converging uniformly to a function f .
 (a) If \mathcal{D}_n is the set of discontinuities of f_n and \mathcal{D} is the set of discontinuities of f , show that $\mathcal{D} \subseteq \cup_{n=1}^{\infty} \cap_{j=n}^{\infty} \mathcal{D}_j$.
 (b) Suppose that for some finite k , each f_n is discontinuous at most at k points. What can you say about the set of discontinuities of f ?
7. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers.
 (a) Define a sequence (f_n) of functions on $[-\pi, \pi]$ by $f_n(x) = \sum_{m=1}^n a_m \sin mx$. Show that each f_n is differentiable with $f'_n(x) = \sum_{m=1}^n m a_m \cos mx$.
 (b) Show that $f(x) = \sum_{m=1}^{\infty} a_m \sin mx$ defines a continuous function on $[-\pi, \pi]$, but that the series $\sum_{m=1}^{\infty} m a_m \cos mx$ need not converge.
8. Show that, for any $x \in X = \mathbb{R} \setminus \{1, 2, 3, \dots\}$, the series $\sum_{m=1}^{\infty} (x - m)^{-2}$ converges. Define $f: X \rightarrow \mathbb{R}$ by $f(x) = \sum_{m=1}^{\infty} (x - m)^{-2}$, and for $n = 1, 2, 3, \dots$, define $f_n: X \rightarrow \mathbb{R}$ by $f_n(x) = \sum_{m=1}^n (x - m)^{-2}$. Does the sequence (f_n) converge uniformly to f on X ? Is f continuous?

9. Let a_n be real numbers such that $\sum_{n=0}^{\infty} a_n$ converges.
- (a) Show that $\sum_{n=1}^{\infty} a_n x^n$ converges for $x \in (-1, 1)$. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, show that f is differentiable on $(-1, 1)$.
- (b)* Show that f extends to $(-1, 1]$ as a continuous function with $f(1) = \sum_{n=0}^{\infty} a_n$. (Hint: start by showing that $f(x) = (1-x) \sum_{n=0}^{\infty} s_n x^n$ for $|x| < 1$, where $s_n = \sum_{j=0}^n a_j$.) Show that, for each $r \in (-1, 1)$, the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[r, 1]$. Must the one-sided derivative $f'(1)$ exist?
10. Is there a real power series with radius of convergence 1 that converges uniformly on $(-1, 1)$?
11. Which of the following functions $f: [0, \infty) \rightarrow \mathbb{R}$ are (a) uniformly continuous; (b) bounded?
- (i) $f(x) = \sin x^2$; (ii) $f(x) = \inf\{|x - n^2| : n \in \mathbb{N}\}$; (iii) $f(x) = (\sin x^3)/(x + 1)$.
12. Show that if (f_n) is a sequence of uniformly continuous, real-valued functions on \mathbb{R} , and if $f_n \rightarrow f$ uniformly, then f is uniformly continuous. Give an example of a sequence of uniformly continuous, real-valued functions (f_n) on \mathbb{R} such that f_n converges pointwise to a function f which is continuous but not uniformly continuous.
13. Suppose that $f: [0, \infty) \rightarrow \mathbb{R}$ is continuous, and that $f(x)$ tends to a (finite) limit as $x \rightarrow \infty$. Must f be uniformly continuous on $[0, \infty)$? Give a proof or a counterexample as appropriate.
14. Let f be a differentiable, real-valued function on \mathbb{R} , and suppose that f' is bounded. Show that f is uniformly continuous. Let $g: [-1, 1] \rightarrow \mathbb{R}$ be the function defined by $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $g(0) = 0$. Show that g is differentiable, but that its derivative is unbounded. Is g uniformly continuous?
15. Let f be a bounded real-valued Riemann integrable functions on $[0, 1]$.
- (a) Must there exist a sequence (f_n) of continuous functions on $[0, 1]$ such that $f_n \rightarrow f$ uniformly on $[0, 1]$?
- (b)* Must there exist a sequence (f_n) of continuous functions on $[0, 1]$ such that $\int_0^1 |f_n(x) - f(x)| dx \rightarrow 0$?
- (c)* Must there exist a sequence (p_n) of polynomials such that $\int_0^1 |p_n(x) - f(x)| dx \rightarrow 0$?
- 16*. Define $\varphi(x) = |x|$ for $x \in [-1, 1]$ and extend φ to \mathbb{R} by requiring that $\varphi(x + 2) = \varphi(x)$.
- (i) Show that $|\varphi(s) - \varphi(t)| \leq |s - t|$ for all s and t .
- (ii) Define $f(x) = \sum_{n=0}^{\infty} (\frac{3}{4})^n \varphi(4^n x)$. Prove that f is well-defined (i.e. the series converges for each x) and that f is continuous.
- (iii) Fix a real number x and positive integer m . Put $\delta_m = \pm \frac{1}{2} 4^{-m}$, where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x + \delta_m)$. Prove that

$$\left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| \geq \frac{1}{2}(3^m + 1).$$

Conclude that f is not differentiable at x . Hence there exists a real continuous function on the real line which is nowhere differentiable.