Analysis II **Michaelmas 2016**

Example Sheet 4

1. (a) Is the set $(1, 2]$ an open subset of $\mathbb{R}$ with the usual metric? Is it closed? What if we replace $\mathbb{R}$ with the space $[0, 2]$, the space $(1, 3)$, or the space $(1, 2]$, in each case with the metric inherited from $\mathbb{R}$?

   (b) Let X be a set equipped with the discrete metric, and let Y be any metric space. Describe all open subsets of X, closed subsets of X, sequentially compact subsets of X, Cauchy sequences in X, continuous functions $f : X \to Y$, and continuous functions $g : Y \to X$.

2. For each of the following sets X, determine whether or not the given function d defines a metric on X. In each case where it does define a metric, describe the open ball $B_\epsilon(x)$ for $x \in X$ and ϵ small.

   (a) $X = \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = \min\{|x_1 - y_1|, |x_2 - y_2|, \ldots |x_n - y_n|\}$

   (b) $X = \mathbb{Z}$, $d(x, x) = 0$, and $d(x, y) = 2^n$ where $x - y = 2^n a$ with n a non-negative integer and a an odd integer.

   (c) $X = \{f : \mathbb{N} \to \mathbb{N}\}$, $d(f, f) = 0$, and $d(f, g) = 2^{-n}$, where n is the smallest natural number such that $f(n) \neq g(n)$.

   (d) $X = \mathbb{C}$, $d(z, w) = |z - w|$ if z and w lie on the same line through the origin, $d(z, w) = |z| + |w|$ otherwise.

3. If X is a metric space and $Y \subset X$, we say Y is bounded if there is a constant M such that $d(y_1, y_2) \leq M$ for all $y_1, y_2 \in Y$. Suppose that every closed bounded subset of X is sequentially compact. Must X be complete?

4. Show that the map $f : [0, 1] \to \mathbb{R}$ given by $f(t) = t \sin \frac{1}{t}$ for $t > 0$, $f(0) = 0$, is uniformly continuous but not Lipshitz.

5. Use the contraction mapping theorem to show that the equation $x = \cos x$ has a unique real solution. Find this solution to some reasonable accuracy using a calculator (remember to work in radians) and justify the claimed accuracy of your approximation.

6. Let X be a complete metric space. Suppose $f : X \to X$ is a contraction map and $g : X \to X$ commutes with f, *i.e.* $f \circ g = g \circ f$. Show that g has a fixed point.

7. Given an example of a non-empty complete metric space X and a function $f : X \to X$ satisfying $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$, but for which f has no fixed point. If X is sequentially compact, show that such an f must have a fixed point.

8. Suppose X and Y are metric spaces. A map $f : X \to Y$ is an *isometric embedding* if $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.

   (a) Show that an isometric embedding is injective.

(b) Suppose that X is sequentially compact and that $f : X \to X$ is an isometric embedding. Show that X is surjective. (Hint: if $x \notin f(X)$, show that $(f^n(x))$ has no convergent subsequence.

(c) Show that the statement in (b) does not hold if "sequentially compact" is replaced by "complete."

(d) Let X be a bounded metric space and let V be the vector space of bounded continuous functions $f : X \to \mathbb{R}$, equipped with the uniform norm. Show that there is an isometric embedding of X into V. (Thus, up to isometry, every bounded metric space is a subspace of a normed space.)

9. Consider the set $C_a = \{(x, y) \in \mathbb{R}^2 \,|\, x^4 + 4x = y^5 + 5ay\}$. Show that there is a unique $a_0 \in \mathbb{R}$ for which C_{a_0} is singular. Sketch C_a for $a < a_0$, $a = a_0$ and $a > a_0$.

10. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Suppose that there is some constant $\mu < 1$ such that $\|Df|_{\mathbf{x}} - I\|_{op} \leq \mu$ for all $\mathbf{x} \in \mathbb{R}^n$. If U is open in $\mathbb{R}^n$, show that $f(U)$ is open in $\mathbb{R}^n$. Show that $\|\mathbf{x} - \mathbf{y}\| \leq (1 - \mu)^{-1}\|f(\mathbf{x}) - f(\mathbf{y})\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Deduce that f is injective and that $f(\mathbb{R}^n)$ is a closed subset of $\mathbb{R}^n$. Conclude that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism.

11. Give an example of a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f'(0) > 0$ such that $f|_{(-\epsilon,\epsilon)}$ is not injective for any $\epsilon > 0$.

12. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function satisfying $\rho(\mathbf{y}) = 1$ for $\|\mathbf{y}\| \leq R$ and $\rho(\mathbf{y}) = 0$ for $\|\mathbf{y}\| \geq R+1$. Suppose $V \in C^1(\mathbb{R}^n)$ and that $\mathbf{y}_0 \in \mathbb{R}^n$ with $\|\mathbf{y}_0\| < R$. How are the solutions to the equations (a) $\mathbf{y}'(t) = V(\mathbf{y}(t))$, subject to $\mathbf{y}(0) = \mathbf{y}_0$ and (b) $\mathbf{y}'(t) = \rho(\mathbf{y}(t))V(\mathbf{y}(t))$ subject to $\mathbf{y}(0) = \mathbf{y}_0$ related?

13. (a) For any $\alpha \in \mathbb{R}$, show that the function $\|\cdot\|_{\infty,\alpha} : C[0, R] \to \mathbb{R}$ given by $\|f\|_{\infty,\alpha} = \|e^{-\alpha x} f\|_\infty$ defines a norm on $C[0, R]$ and that this norm is Lipshitz equivalent to $\|\cdot\|_\infty$.

(b) Now suppose that $V : \mathbb{R}^2 \to \mathbb{R}$ is continuous, and Lipshitz in the second variable. Consider the map $T : C[0, R] \to C[0, R]$ given by

$$(T(f))(t) = y_0 + \int_0^t V(s, f(s))\, ds$$

Show that T is a contraction map with respect to $\|\cdot\|_{\infty,\alpha}$ for some α. Deduce that the differential equation $f'(t) = V(t, f(t))$ has a unique solution on $[0, R]$ satisfying $f(0) = y_0$, and hence that this equation has a unique solution on $[0, \infty)$ satisfying $f(0) = y_0$.

14. (a) Show that for small values of x, y, z and w, the set of solutions to the equations

$$\sin xz + \cos yw = e^z$$
$$\cos yz + \sin xw = e^w$$

consists of points of the form $(x, y, F(x, y), G(x, y))$, where $F, G : B_\epsilon(\mathbf{0}) \to \mathbb{R}$ are C^1 functions.

(b) Deduce that for small values of t, the system of differential equations

$$\sin y_1 y_1' + \cos y_2 y_2' = e^{y_1'}$$
$$\cos y_2 y_1' + \sin y_1 y_2' = e^{y_2'}$$

has a unique solution $\mathbf{y}(t) = (y_1(t), y_2(t))$ satisfying $\mathbf{y}(0) = \mathbf{0}$.