Example Sheet 3

1. Consider the map $f : \mathbb{R}^6 \to \mathbb{R}^3$ defined by $f(\mathbf{x}, \mathbf{y}) = \mathbf{x} \times \mathbf{y}$ (*i.e.* the usual cross product of vectors in $\mathbb{R}^3$.) Prove directly from the definition that f is differentiable and express its derivative at $(\mathbf{x}, \mathbf{y})$ first as a linear map and then as a matrix.

2. At which points of $\mathbb{R}^2$ are the following functions $\mathbb{R}^2 \to \mathbb{R}$ differentiable?

   (a) $f(x, y) = xy|x - y|$.

   (b) $f(x, y) = xy/\sqrt{x^2 + y^2}$ for $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.

   (c) $f(x, y) = xy \sin 1/x$ for $x \neq 0$, $f(0, y) = 0$.

3. Show that the function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{v}) = \|\mathbf{v}\|_2$ is differentiable at all nonzero $\mathbf{v} \in \mathbb{R}^n$. (Hint: first show that $\mathbf{v} \mapsto \|\mathbf{v}\|_2^2$ is differentiable.) At which points in $\mathbb{R}^2$ are the functions $\|\cdot\|_1$ and $\|\cdot\|_\infty$ differentiable?

4. Let $f(x, y) = x^2 y/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show that f is continuous at $(0, 0)$ and that it has directional derivatives in all directions there. Is f differentiable at $(0, 0)$?

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function, and let $g(x) = f(x, c - x)$, where c is a constant. Show that $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and find its derivative a) directly from the definition and b) by using the chain rule. Deduce that if $D_2 f = D_1 f$ everywhere in $\mathbb{R}^2$, then $f(x, y) = h(x + y)$ for some differentiable function $h : \mathbb{R} \to \mathbb{R}$.

6. We work in $\mathbb{R}^n$ with the usual inner product and $\|\cdot\| = \|\cdot\|_2$. Consider the map $f : \mathbb{R}^n \to \mathbb{R}^n$ given by $f(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$ for $\mathbf{x} \neq \mathbf{0}$ and $f(\mathbf{0}) = \mathbf{0}$. Show that f is differentiable except at $\mathbf{0}$ and
$$Df|_{\mathbf{x}}(\mathbf{v}) = \frac{\mathbf{v}}{\|\mathbf{x}\|} - \langle \mathbf{x}, \mathbf{v} \rangle \frac{\mathbf{x}}{\|\mathbf{x}\|^3}.$$
Verify that $Df|_{\mathbf{x}}(\mathbf{v})$ is orthogonal to $\mathbf{x}$ and explain geometrically why this is the case.

7. Suppose $F : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. If the directional derivative $D_{\mathbf{v}} F|_{\mathbf{x}}$ exists for all $\mathbf{v} \in \mathbb{R}^n$ and is a linear function of $\mathbf{v}$, must F be differentiable at $\mathbf{x}$?

8. Let $f(x, y) = xy(x^2 - y^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show that

   (a) $f, D_1 f$, and $D_2 f$ are continuous in $\mathbb{R}^2$.

   (b) $D_{12} f$ and $D_{21} f$ exist at every point in $\mathbb{R}^2$ and are continuous except at $(0, 0)$.

   (c) $D_{12} f|_{(0,0)} \neq D_{21} f|_{(0,0)}$.

9. Let $V = M_{n \times n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$, and let $U \subset V$ be an open subset. Given $f, g : U \to V$, define $fg : U \to V$ by $fg(X) = f(X)g(X)$ (matrix multiplication). If f and g are differentiable, show that fg is differentiable, and that $D(fg)|_X(A) = Df|_X(A)g(X) + f(X)Dg|_X(A)$. Now let $U \subset V$ be the set of invertible matrices, and define $g : U \to V$ by $g(X) = X^{-1}$. Show that g is differentiable and compute its derivative.

10. Let $V = M_{n\times n}(\mathbb{R})$ as above. By considering $\det(I + A)$ as a polynomial in the entries of A, show that the function $\det : V \to \mathbb{R}$ is differentiable at the identity matrix I and that its derivative there is the function $A \mapsto \operatorname{tr} A$. Hence show that det is differentiable at any invertible matrix X, with derivative $A \mapsto \det X \operatorname{tr}(X^{-1}A)$. Compute the second derivative of det at I as a bilinear map $V \times V \to \mathbb{R}$, and verify it is symmetric.

11. a) Let $V = M_{n\times n}(\mathbb{R})$, and define $f : V \to V$ by $f(X) = X^3$. Find the Taylor series for $f(X + A)$ centered at X. b)* Let $U \subset V$ be the set of invertible matrices, and define $g : U \to U$ by $g(X) = X^{-1}$. Find the Taylor series for $g(I + A)$ centered at I.

12. * A *critical point* of a differentiable function $F : \mathbb{R}^n \to \mathbb{R}$ is a point $\mathbf{x} \in \mathbb{R}^n$ for which $DF|_{\mathbf{x}} = 0$. Suppose that $\mathbf{x}$ is a critical point of F such that the second derivative $D^2F|_{\mathbf{x}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a nondegenerate quadratic form. (That is, for any $\mathbf{v} \neq \mathbf{0}$ in $\mathbb{R}^n$, there is some $\mathbf{w}$ with $D^2F|_{\mathbf{x}}(\mathbf{v}, \mathbf{w}) \neq 0$.) Show that F attains a local maximum at $\mathbf{x}$ if and only if $D^2F|_{\mathbf{x}}$ is negative definite. (That is, $D^2F|_{\mathbf{x}}(\mathbf{v}, \mathbf{v}) < 0$ for all $\mathbf{v} \neq \mathbf{0}$.)

13. * Let $U \subset \mathbb{R}^2$ be an open set containing the rectangle $[a, b] \times [c, d]$. Suppose that $g : U \to \mathbb{R}$ is continuous and that $D_2 g$ exists and is continuous on U. Set

$$G(y) = \int_a^b g(x, y)\, dx.$$

Show that G is differentiable on (c, d) with derivative

$$G'(y) = \int_a^b D_2 g(x, y)\, dx.$$

(Hint: $D_2 g$ is uniformly continuous on $[a, b] \times [c, d]$.)

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