Analysis II **Michaelmas 2016**

Example Sheet 2

1. Let V be a normed space.

   (a) If $(\mathbf{v}_n) \to \mathbf{v} \in V$, show that any subsequence of $(\mathbf{v}_n)$ converges to $\mathbf{v}$.

   (b) If $(\mathbf{v}_n)$ is a Cauchy sequence in V, show that it is bounded.

2. Suppose $(f_n) \to f$ pointwise, where $f_n, f \in C[0,1]$.

   (a) If $(f_n) \to f$ uniformly and (x_m) is a sequence of points in $[0,1]$ converging to x, show that $(f_n(x_n)) \to f(x)$.

   (b) If (f_n) does not converge uniformly, show that there is a convergent sequence $(x_m) \to x \in [0,1]$ such that $(f_n(x_n))$ does not converge to $f(x)$. (Hint: Bolzano-Weierstrass.)

3. If A and B are subsets of $\mathbb{R}^n$, let $A + B = \{a + b \,|\, a \in A, b \in B\}$. Show that if A and B are both closed and one of them is bounded, then $A + B$ is closed. Give an example in $\mathbb{R}$ to show that the boundedness condition cannot be omitted. If A and B are both open, is $A + B$ necessarily open? Justify your answer.

4. Let $(V, \|\cdot\|)$ be a complete normed space, and let W be a linear subspace of V. Show that $(W, \|\cdot\|)$ is complete if and only W is a closed subset of V. Which of the following vector spaces of functions on $\mathbb{R}$ are complete with respect to the uniform norm?

   (a) The space $C_b(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$.

   (b) The space $C_0(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$.

   (c) The space $C_c(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $|x|$ sufficiently large.

5. Let $(V, \|\cdot\|)$ be a normed space, and let $(\mathbf{v}_n)$ be a sequence in V.

   (a) If V is complete, show that $\sum_{n=1}^{\infty} \mathbf{v}_n$ converges whenever $\sum_{n=1}^{\infty} \|\mathbf{v}_n\|$ converges.

   (b)* If $\sum_{n=1}^{\infty} \mathbf{v}_n$ converges whenever $\sum_{n=1}^{\infty} \|\mathbf{v}_n\|$ converges, show that V is complete. (Hint: If $(\mathbf{v}_n)$ is Cauchy, there is a subsequence $(\mathbf{v}_{n_i})$ such that $\sum_{i=1}^{\infty} \|\mathbf{v}_{n_{i+1}} - \mathbf{v}_{n_i}\|$ converges.)

6. Which of the following functions $f : [0,\infty) \to \mathbb{R}$ are uniformly continuous?

   (a) $f(x) = \sin x^2$ (b) $f(x) = \inf\{|x - n^2| \,|\, n \in \mathbb{Z}\}$ (c) $(\sin x^3)/(x+1)$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and suppose that f' is bounded. Show that f is uniformly continuous. Let $g : [-1,1] \to \mathbb{R}$ be given by $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $g(0) = 0$. Show that g is differentiable, but that its derivative is unbounded. Is g uniformly continuous?

8. Suppose that $f : [0,\infty) \to \mathbb{R}$ is continuous, and that $f(x)$ tends to a finite limit as $x \to \infty$. Must f be uniformly continuous on $[0,\infty)$? Give a proof or a counterexample.

9. Let ℓ^1 be the set of real sequences (x_n) such that $\sum_{n=1}^{\infty}|x_n|$ is convergent. Show that with addition and scalar multiplication defined termwise, ℓ^1 is a vector space. Define $\|\cdot\|_1 : \ell^1 \to \mathbb{R}$ by $\|(x_n)\|_1 = \sum_{n=1}^{\infty}|x_n|$. Show that $\|\cdot\|_1$ is a norm on ℓ^1, and that $(\ell^1, \|\cdot\|_1)$ is complete. Is $\overline{B}_1(0, \|\cdot\|_1)$ sequentially compact?

10. Let ℓ^∞ be the vector space of bounded real sequences $\mathbf{x} = (x_n)$, with the norm given by $\|(x_n)\| = \sup |x_n|$. Let $(\mathbf{x}_k)$ be a bounded sequence in ℓ^∞, and let $x_{i,k} \in \mathbb{R}$ be the ith entry in the sequence $\mathbf{x}_k$. Show that $(\mathbf{x}_k)$ has a subsequence $(\mathbf{x}_{k_j})$ such that for each fixed value of i, the sequence (x_{i,k_j}) converges as $j \to \infty$. (Hint: use a diagonal argument.) Must $\mathbf{x}_{k_j}$ converge in ℓ^∞? How is this example related to the notions of pointwise and uniform convergence?

11. Show that $A = \{(x_n) \in \ell^1 \,|\, |x_n| \leq 1/n^2 \text{ for all } n\}$ is a sequentially compact subset of $(\ell^1, \|\cdot\|_1)$, but that $B = \{(x_n) \in \ell^1 \,|\, |x_n| \leq 1/n \text{ for all } n\}$ is not.

12. Let V be a complete normed space, and let $\mathcal{B}(V, V)$ be the space of continuous linear maps from V to itself. Show that $\mathcal{B}(V, V)$ is complete with respect to the operator norm defined in problem 6 of the first example sheet. Show that if $\phi \in \mathcal{B}(V, V)$ satisfies $\|\phi\| < 1$, then $I - \phi$ is invertible, where I is the identity map. (Hint: consider $I + \phi + \phi^2 + \ldots$) Deduce that the set of invertible maps is an open subset of $\mathcal{B}(V, V)$.

13.* (*A space-filling curve*) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the following properties: $f(t) = 0$ for $t \in [0, 1/3]$; $f(t) = 1$ for $t \in [2/3, 1]$; $0 \leq f(t) \leq 1$ for all $t \in \mathbb{R}$; and $f(t + 2) = f(t)$ for all $t \in \mathbb{R}$. Define $\Phi(t) = (x(t), y(t))$, where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \quad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$$

Prove that Φ is continuous and maps the unit interval $I = [0, 1]$ onto the unit square $I^2 \subset \mathbb{R}^2$. (Hint: show that $\Phi(I)$ contains all points of the form $(a/2^n, b/2^n)$, where $a, b \in \mathbb{Z}$, $0 \leq a, b \leq 2^n$.)

14.* Suppose C is a sequentially compact subset of a normed space V. If $\{U_\alpha \,|\, \alpha \in A\}$ is a family of open subsets of V, we say that the U_α cover C if $C \subset \cup_{\alpha \in A} U_\alpha$.

   (a) Given $\epsilon > 0$, show that there is a finite set of points $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset C$ such that $\{B_\epsilon(\mathbf{v}_1), \ldots, B_\epsilon(\mathbf{v}_n)\}$ cover C.

   (b) Show that there exists some $\epsilon > 0$ such that for every $\mathbf{v} \in C$, $B_\epsilon(\mathbf{v}) \subset U_\alpha$ for some α. Deduce that there is a finite set $\{\alpha_1, \ldots, \alpha_n\} \subset A$ such that $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ cover C.

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