ANALYSIS II—EXAMPLES 3 Mich. 2015

Please email comments, corrections to: `n.wickramasekera@dpmms.cam.ac.uk`.

1. Quickies: (a) Use the equivalence of norms on a finite dimensional vector space to show that for each n, there is a constant C such that the following holds: for every polynomial p of degree $\leq n$ there is $x_0 \in [0, 1/n]$ such that $|p(x)| \leq C|p(x_0)|$ for every $x \in [0, 1]$.

(b) If (X, d) is a metric space and A is a non-empty subset of X, show that the distance from $x \in X$ to A defined by $\rho(x) = \inf_{y \in A} d(x, y)$ is a Lipschitz function on X with Lipschitz constant equal to 1.

(c) If every closed, bounded subset of a metric space X is compact, must X be complete?

(d) If every closed proper subset of a metric space X is complete relative to the induced metric, must X be complete?

(e) If (x_n), (y_n) are Cauchy sequences in a metric space (X, d), show that $(d(x_n, y_n))$ is convergent (in $\mathbb{R}$).

2. (a) Is the set $(1, 2]$ an open subset of the metric space $\mathbb{R}$ with the metric $d(x, y) = |x - y|$? Is it closed? What if we replace the metric space $\mathbb{R}$ with the space $[0, 2]$, the space$(1, 3)$ or the space $(1, 2]$, in each case with the metric d?

(b) Let X be a set equipped with the discrete metric, and Y any metric space. Describe all open subsets of X, closed subsets of X, sequentially compact subsets of X, Cauchy sequences in X, continuous functions $X \to Y$ and continuous functions $Y \to X$.

3. For each of the following sets X, determine whether or not the given function d defines a metric on X. In each case where the function does define a metric, describe the open ball $B_\varepsilon(x)$ for $x \in X$ and $\varepsilon > 0$ small.

(i) $X = \mathbb{R}^n$; $d(x, y) = \min\{|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|\}$.

(ii) $X = \mathbb{Z}$; $d(x, x) = 0$, and, for $x \neq y$, $d(x, y) = 2^n$ where $x - y = 2^n a$ with n a non-negative integer and a an odd integer.

(iii) X is the set of functions from $\mathbb{N}$ to $\mathbb{N}$; $d(f, f) = 0$, and, for $f \neq g$, $d(f, g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

(iv) $X = \mathbb{C}$; $d(z, w) = |z - w|$ if z and w lie on the same line through the origin, $d(z, w) = |z| + |w|$ otherwise.

4. Let (X, d) be a metric space.

(a) Show that the union of any collection of open subsets of X must be open (regardless of whether the collection is finite, countable or uncountable), and that the intersection of any finite collection of open subsets is again open. Formulate and prove similar properties about the closed subsets of X.

(b) Let E be a subset of X. Show that there is a unique largest open subset E^o of X contained in E, i.e. a unique open subset E^o of X such that that $E^o \subseteq E$ and if G is any open subset of X with $G \subseteq E$ then $G \subseteq E^o$. E^o is called the *interior* of E in X. Show also that there is a unique smallest closed subset $\overline{E}$ of X containing E, i.e. a unique closed subset $\overline{E}$ of X with $E \subseteq \overline{E}$ and if F is any closed subset of X with $E \subseteq F$ then $\overline{E} \subseteq F$. $\overline{E}$ is called the *closure* of E in X.

(c) Show that

$$E^o = \{x \in X \, : \, B_\epsilon(x) \subset E \quad \text{for} \quad \text{some} \quad \epsilon > 0\}$$

and that

$$\overline{E} = \{x \in X \, : \, x_n \to x \quad \text{for} \quad \text{some} \quad \text{sequence} \quad (x_n) \quad \text{in} \quad E\}.$$

5. Let V be a normed space, $x \in V$ and $r > 0$. Prove that the closure of the open ball $B_r(x)$ is the closed ball $D_r(x) = \{y \in V : \|x - y\| \leq r\}$. Give an example to show that, in a general metric space (X, d), the closure of the open ball $B_r(x)$ need not be the closed ball $D_r(x) = \{y \in X : d(x, y) \leq r\}$.

6. In lectures we proved that if E is a closed, bounded subset of $\mathbb{R}^n$ with the Euclidean metric, then any continuous function on E has bounded image. Prove the converse: if E is a subset of $\mathbb{R}^n$ with the Euclidean metric and if every continuous function $f : E \to \mathbb{R}$ has bounded image, then E is closed and bounded.

7. Each of the following properties/notions makes sense for an arbitrary metric spaces X. Which are topological (i.e. dependent only on the collection of open subsets of X and not on the metric generating the open subsets)? Justify your answers.

(i) boundedness of a subset of X.

(ii) closed-ness of a subset of X.

(iii) notion that a subset of X is closed *and* bounded.

(iv) total boundedness of X; that is, the property that for every $\epsilon > 0$, there is a finite set $F \subset X$ such that the union of open balls with centres in F and radius ϵ is X.

(v) completeness of X.

(vi) total boundedness *and* completeness of X.

8. Use the Contraction Mapping Theorem to show that the equation $\cos x = x$ has a unique real solution. Find this solution to some reasonable accuracy using a calculator (remember to work in radians!), and justify the claimed accuracy of your approximation.

9. Let $I = [0, R]$ be an interval and let $C(I)$ be the space of continuous functions on I. Show that, for any $\alpha \in \mathbb{R}$, we may define a norm by $\|f\|_\alpha = \sup_{x\in I} |f(x)e^{-\alpha x}|$, and that the norm $\|\cdot\|_\alpha$ is Lipschitz equivalent to the uniform norm $\|f\| = \sup_{x\in I} |f(x)|$.
Now suppose that $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, and Lipschitz in the second variable. Consider the map T from $C(I)$ to itself sending f to $y_0 + \int_0^x \phi(t, f(t))dt$. Give an example to show that T need not be a contraction under the uniform norm. Show, however, that T is a contraction under the norm $\|\cdot\|_\alpha$ for some α, and hence deduce that the differential equation $f'(x) = \phi(x, f(x))$ has a unique solution on I satisfying $f(0) = y_0$.

10. Let (X, d) be a non-empty complete metric space. Suppose $f\colon X \to X$ is a contraction and $g\colon X \to X$ is a function which commutes with f, i.e. such that $f(g(x)) = g(f(x))$ for all $x \in X$. Show that g has a fixed point. Must this fixed point be unique?

11. Give an example of a non-empty complete metric space (X, d) and a function $f\colon X \to X$ satisfying $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, but such that f has no fixed point. Suppose now that X is a non-empty closed bounded subset of $\mathbb{R}^n$ with the Euclidean metric. Show that in this case f must have a fixed point. If $g\colon X \to X$ satisfies $d(g(x), g(y)) \le d(x, y)$ for all $x, y \in X$, must g have a fixed point?

12.* Show that it is not possible to obtain, starting from an arbitrary set $X \subseteq \mathbb{R}^n$ and repeatedly applying the operations $(\cdot)^o$ (interior) and $\overline{(\cdot)}$ (closure), more than seven distinct sets (including X itself). Give an example in $\mathbb{R}$ where seven sets are obtained.

13.* Let (X, d) be a non-empty complete metric space and let $f\colon X \to X$ be a function such that for each positive integer n we have
(i) if $d(x, y) < n + 1$ then $d(f(x), f(y)) < n$; and
(ii) if $d(x, y) < 1/n$ then $d(f(x), f(y)) < 1/(n + 1)$.
Must f have a fixed point?

14.* Let K be a closed bounded subset of $\mathbb{R}$ and $p \in K$. Construct a metric d on $K_1 = K \setminus \{p\}$ such that (K_1, d) is complete and the topology generated by d on K_1 is the same as the topology generated by the Euclidean metric on K_1.

15.* It is a consequence of the *Baire category theorem* (which you can learn about in the Linear Analysis course next year for example) that if f is the pointwise limit of a sequence of continuous functions $f_n : [a, b] \to \mathbb{R}$, then f has a point of continuity (and hence in fact a dense subset of $[a, b]$ of continuity points). Taking this fact for granted, and considering the family of functions $f_{n,m}(x) = (\cos n!\pi x)^{2m}$, $n, m \in \mathbb{N}$, show that pointwise convergence of continuous functions on an interval $[a, b]$ is not metrizable. That is to say, show that there is no metric d on the set of continuous functions $f : [a, b] \to \mathbb{R}$ such that pointwise convergence of sequences of functions in this set is equivalent to convergence with respect to d.