ANALYSIS II—EXAMPLES 2 Mich. 2015

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1. Quickies: (a) Describe all continuous functions $f : [0,1] \to \mathbb{R}^n$ satisfying $\| \int_0^1 f\| = \int_0^1 \|f\|$.

(b) Show that two norms $\|\cdot\|$, $\|\cdot\|'$ on a vector space V are Lipschitz equivalent if and only if there exist numbers $r, R > 0$ such that $B_r \subseteq B'_1 \subseteq B_R$, where for $\rho > 0$, $B_\rho = \{x \in V : \|x\| < \rho\}$ and $B'_\rho = \{x \in V : \|x\|' < \rho\}$.

(c) If $(V, \|\cdot\|)$ is a normed space and $\varphi : V \to \mathbb{R}$ is a linear functional, show that $\|\cdot\| + |\varphi(\cdot)|$ defines a norm on V, and that this norm is not Lipschitz equivalent to $\|\cdot\|$ if φ is not continuous.

(d) If a Cauchy sequence (x_n) in a normed space has a subsequence converging to an element x, show that the whole sequence (x_n) converges to x.

2. Let $(x^{(m)})$ and $(y^{(m)})$ be sequences in $\mathbb{R}^n$ converging to x and y respectively. Show that $x^{(m)} \cdot y^{(m)}$ converges to $x \cdot y$. Deduce that if $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are continuous at $x \in \mathbb{R}^n$, then so is the pointwise scalar product $f \cdot g : \mathbb{R}^n \to \mathbb{R}$.

3. (a) Show that $\|f\|_1 = \int_0^1 |f|$ defines a norm on the vector space $C\big([0,1]\big)$. Is it Lipschitz equivalent to the uniform norm? Is $C\big([0,1]\big)$ with norm $\|\cdot\|_1$ complete?

(b) Let $R\big([0,1]\big)$ denote the vector space of all bounded Riemann integrable functions on $[0,1]$. Does $\|f\|_1 = \int_0^1 |f|$ define a norm on $R\big([0,1]\big)$? If so, is $R\big([0,1]\big)$ complete with this norm? What if we replace $\|\cdot\|_1$ with $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$?

4. (a) Let $C^1([0,1])$ be the vector space of real continuous functions on $[0,1]$ with continuous first derivatives. Define functions $\alpha, \beta, \gamma, \delta : C^1([0,1]) \to \mathbb{R}$ by $\alpha(f) = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|$; $\beta(f) = \sup_{x\in[0,1]} (|f(x)| + |f'(x)|)$; $\gamma(f) = \sup_{x\in[0,1]} |f(x)|$; $\delta(f) = \sup_{x\in[0,1]} |f'(x)|$. Which of these define norms on $C^1([0,1])$? Out of those that define norms, which pairs are Lipschitz equivalent?

(b) Let $C^1_c([0,1])$ be the set of functions $f \in C^1([0,1])$ such that $f(x) = 0$ for x in some neighborhood of the end points 0 and 1. Verify that $C^1_c([0,1])$ is a vector space. How would your answers in (a) change if we replace $C^1([0,1])$ by $C^1_c([0,1])$?

5. Which of the following subsets of $\mathbb{R}^2$ with the Euclidean norm are open? Which are closed? (And why?)

(i) $\{(x,0) : 0 \le x \le 1\}$;
(ii) $\{(x,0) : 0 < x < 1\}$;
(iii) $\{(x,y) : y \ne 0\}$;
(iv) $\{(x,y) : x \in \mathbb{Q} \quad \text{or} \quad y \in \mathbb{Q}\}$;
(v) $\{(x,y) : y = nx \quad \text{for some} \quad n \in \mathbb{N}\} \cup \{(x,y) : x = 0\}$;
(vi) $\{(x, f(x)) : x \in \mathbb{R}\}$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

6. Is the set $\{f : f(1/2) = 0\}$ closed in the space $C\big([0,1]\big)$ with the uniform norm? What about the set $\{f : \int_0^1 f = 0\}$? In each case, does the answer change if we replace the uniform norm with the norm $\|\cdot\|_1$ defined in Q3?

7. Which of the following functions f are continuous?

(i) The linear map $f\colon \ell^\infty \to \mathbb{R}$ defined by $f(x) = \sum_{n=1}^\infty x_n/n^2$;
(ii) The identity map from the space $C\big([0,1]\big)$ with the uniform norm $\|\cdot\|$ to the space $C\big([0,1]\big)$ with the norm $\|\cdot\|_1$ defined in Q3;
(iii) The identity map from $C\big([0,1]\big)$ with the norm $\|\cdot\|_1$ to $C\big([0,1]\big)$ with the uniform norm $\|\cdot\|$;
(iv) The linear map $f\colon \ell^0 \to \mathbb{R}$ defined by $f(x) = \sum_{i=1}^\infty x_i$, where ℓ^0 has norm $\|\cdot\|_\infty$. (ℓ^0 is the space of real sequences (x_k) such that $x_k = 0$ for all but a finite number of k.)

8. Is it possible to find uncountably many norms on $C([0,1])$ such that no two are Lipschitz equivalent?

9. Let ℓ^1 denote the set of real sequences (x_n) such that $\sum_{n=1}^\infty |x_n|$ is convergent. Show that, with addition and scalar multiplication defined termwise, ℓ^1 is a vector space. Define $\|\cdot\|_1\colon \ell^1 \to \mathbb{R}$ by $\|x\|_1 = \sum_{n=1}^\infty |x_n|$. Show that $\|\cdot\|_1$ is a norm on ℓ^1, and that $(\ell^1, \|\cdot\|_1)$ is complete.

$10^\star$. Let $(V, \|\cdot\|)$ be a normed space. Show that V is complete if and only if V has the property that for every sequence (x_n) in V with $\sum_{j=1}^\infty \|x_n\|$ convergent, the series $\sum_{n=1}^\infty x_n$ is convergent. (Thus V is complete if and only if every absolutely convergent series in V is convergent.) [Hint: If (x_n) is Cauchy, then there is a subsequence (x_{n_j}) such that $\sum_j \|x_{n_{j+1}} - x_{n_j}\| < \infty$.]

11. Let V be a normed space in which every bounded sequence has a convergent subsequence. (a) Show that this property of V is equivalent to the sequential compactness of the unit sphere $S = \{x \in V : \|x\| = 1\}$. (b) Show that V must be complete. (c)* Show further that V must be finite-dimensional.
[Hint for (c): Start by showing that for every finite-dimensional subspace V_0 of V, there exists $x \in V$ with $\|x + y\| > \|x\|/2$ for each $y \in V_0$.]

12. Let $(x^{(n)})_{n\geq 1}$ be a bounded sequence in ℓ^∞. Show that there is a subsequence $(x^{(n_j)})_{j\geq 1}$ which converges in every coordinate; that is to say, the sequence $(x_i^{(n_j)})_{j\geq 1}$ of real numbers converges for each i. Why does this not show that every bounded sequence in ℓ^∞ has a convergent subsequence?

13. (a) Let $(V, \|\cdot\|)$ be a complete normed space, and let W be a subspace of V. Show that $(W, \|\cdot\|)$ is complete if and only if W is closed in V.
(b) Which of the following vector spaces of functions, taken with the uniform norm, are complete?
(i) The space $C_b(\mathbb{R})$ of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$.
(ii) The space $C_0(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$.
(iii) The space $C_c(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $|x|$ sufficiently large.

$14^\star$. Let $\mathcal{P}$ be the vector space of real polynomials on the unit interval $[0, 1]$. Show that for any infinite set $I \subseteq [0, 1]$, $\|p\|_I = \sup_I |p|$ defines a norm on $\mathcal{P}$. Use this fact to produce an example of a vector space, a sequence in it and two different norms on it such that the sequence converges to different elements in the space with respect to the different norms. (Hint: the Weierstrass approximation theorem may be helpful).
Is it possible to find such a sequence in one of the spaces ℓ^1 or ℓ^2 equipped with two norms, when possible, chosen from the standard norms on the spaces ℓ^1, ℓ^2, ℓ^∞? What about in the space $C([0, 1])$ equipped with two norms chosen from the L^1, L^2, L^∞ norms?

Supplement: A proof of Lebesgue's theorem on the Riemann integral. Let $f : [a, b] \to \mathbb{R}$ be bounded. Recall that Lebesgue's theorem says that f is Riemann integrable on $[a, b]$ if and only if the set $\mathcal{D}_f$ of points in $[a, b]$ where f is discontinuous has Lebesgue measure zero. (By definition, a set $\mathcal{D} \subset \mathbb{R}$ has Lebesgue measure zero if for every $\epsilon > 0$, there is a countable collection of open intervals $I_j = (a_j, b_j)$ such that $\mathcal{D} \subset \cup_{j=1}^\infty I_j$ and $\sum_{j=1}^\infty |I_j| < \epsilon$, where $|I_j| = b_j - a_j$.) As an optional exercise, prove this theorem by completing the outline below. We shall use the notation as in lectures, so $U(P, f)$, $L(P, f)$ denote the upper and lower sums for f relative to a partition P of $[a, b]$.

(a) Show that $y \in \mathcal{D}_f \cap (a, b)$ (i.e. y is an interior discontinuity) if and only if there exists $\epsilon = \epsilon_y > 0$ such that $\sup_I f - \inf_I f > \epsilon$ for every open interval $I \subset [a, b]$ with $y \in I$. Hence $\mathcal{D}_f \cap (a, b) = \cup_{j=1}^\infty E_j$, where $E_j = \{y \in (a, b) : \sup_I f - \inf_I f > j^{-1}$ for every open interval I with $y \in I\}$.

(b) Suppose that f is Riemann integrable. It suffices to show that E_j has Lebesgue measure zero for each j (Why?). Fix j, let $\epsilon > 0$ and choose a partition $P = \{a = a_0 < a_1 < \ldots < a_n = b\}$ such that $U(P, f) - L(P, f) < j^{-1}\epsilon$. Let $K = \{k : E_j \cap (a_k, a_{k+1}) \neq \emptyset\}$. Then $E_j \setminus \{a_0, a_1, \ldots, a_n\} \subset \cup_{k\in K}(a_k, a_{k+1})$. Show that $\sum_{k\in K}(a_{k+1} - a_k) < \epsilon$. Deduce that E_j has Lebesgue measure zero.

(c) Now suppose that $\mathcal{D}_f$ has Lebesgue measure zero. Let $\epsilon > 0$, and choose open intervals $I_j \subset \mathbb{R}$, $j = 1, 2, \ldots$, with $\mathcal{D}_f \subset \cup_{j=1}^\infty I_j$ and $\sum_{j=1}^\infty |I_j| < \epsilon$. Let $F = [a, b] \setminus \cup_{j=1}^\infty I_j$. Show that there exists $\delta > 0$ such that the following holds: $x \in F$, $y \in [a, b]$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. [This is a strengthening of the theorem we proved in lecture that says that a continuous function on a closed, bounded interval (or more generally on a compact metric space) is uniformly continuous, but the same contradiction argument we used in fact works here.] Let $P = \{a = a_0 < a_1 < a_2 \ldots < a_n = b\}$ be any partition of $[a, b]$ such that $a_{j+1} - a_j < \delta$, and let $J = \{j : [a_j, a_{j+1}] \cap F \neq \emptyset\}$. Show that $\sup_{[a_j,a_{j+1}]} f - \inf_{[a_j,a_{j+1}]} f < 2\epsilon$ for each $j \in J$, and that $\cup_{j\notin J}(a_j, a_{j+1}) \subset \cup_{j=1}^\infty I_j$. Conclude that $U(P, f) - L(P, f) < 2(b - a + \sup_{[a,b]} |f|)\epsilon$, and hence that f is Riemann integrable on $[a, b]$.