ANALYSIS II—EXAMPLES 4 Mich. 2014

The questions marked with * are intended as additional. Please email comments, corrections to: `n.wickramasekera@dpmms.cam.ac.uk`.

1. Quickies: (i) Let $F : [0,1] \times \mathbb{R}^m \to \mathbb{R}$ be continuous and $a = (a_0, \ldots, a_{m-1}) \in \mathbb{R}^m$. Suppose that F is uniformly Lipschitz in the $\mathbb{R}^m$ variables near a, i.e. for some constant K and an open subset U of $\mathbb{R}^m$ containing a, $|F(t,x) - F(t,y)| \leq K\|x - y\|$ for all $t \in [0,1]$, $x, y \in U$. Use the Picard–Lindelöf existence theorem for first order ODE systems to show that there is an $\epsilon > 0$ such that, writing $f^{(j)}$ for the jth derivative of f, the mth order initial value problem

$$f^{(m)}(t) = F(t, f(t), f^{(1)}(t), \ldots, f^{(m-1)}(t)) \quad \text{for} \quad t \in [0, \epsilon);$$

$$f^{(j)}(0) = a_j \quad \text{for} \quad 0 \leq j \leq m-1$$

has a unique C^m solution $f : [0, \epsilon) \to \mathbb{R}$.

(ii) Let $f: \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$. If the directional derivatives $D_u f(a)$ exist for all directions $u \in \mathbb{R}^n$ and if $D_u f(a)$ depends linearly on u, does it follow that f is differentiable at a?

(iii) Let $f: \mathbb{R}^n \to \mathbb{R}$, $a \in \mathbb{R}^n$ and suppose that f is differentiable at a. Define the *gradient* of f at a to be the vector $\nabla f(a) = (D_1 f(a), \ldots, D_n f(a))$. Show that $\max\{D_u f(a) : u \in \mathbb{R}^n, \|u\| = 1\} = \|\nabla f(a)\|$, and if $\nabla f(a) \neq 0$, that this maximum is attained when and only when $u = \frac{\nabla f(a)}{\|\nabla f(a)\|}$. What does this say about the rates of change of f at a in different directions?

(iv) Let $f: [a,b] \to \mathbb{R}^2$ be continuous, and differentiable on (a,b). Does there exist $c \in (a,b)$ such tat $f(b) - f(a) = f'(c)(b-a)$?

(v) Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map with $\|Df(x) - I\| \leq 1/2$ for each $x \in \mathbb{R}^n$, where I is the identity map on $\mathbb{R}^n$. Does it follow that f is one-to-one? Does it follow that f is an open mapping, i.e. that f maps open sets to open sets?

2. (a) Let $f = (f_1, \ldots, f_m): \mathbb{R}^n \to \mathbb{R}^m$. Show that f is differentiable at $x \in \mathbb{R}^n$ iff each $f_i: \mathbb{R}^n \to \mathbb{R}$ is differentiable at x, and in this case, $Df(x)(h) = (Df_1(x)(h), \ldots, Df_m(x)(h))$ for each $h \in \mathbb{R}^n$.

(b) Define $f: \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x,y,z) = (e^{x+y+z}, \cos x^2 y)$. Without making use of partial derivatives, show that f is everywhere differentiable and find $Df(a)$ at each $a \in \mathbb{R}^3$.

(c) Find all partial derivatives of f and hence, using appropriate results on partial derivatives, give an alternative proof of the result of (b).

3. Let $\mathcal{M}_n$ be the space of $n \times n$ real matrices. (Note that $\mathcal{M}_n$ can be identified with $\mathbb{R}^{n^2}$.) Define $f: \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^4$. Show that f is differentiable at every $A \in \mathcal{M}_n$, and find $Df(A)$ as a linear map. Show further that f is twice-differentiable at every $A \in \mathcal{M}_n$ and find $D^2 f(A)$ as a bilinear map from $\mathcal{M}_n \times \mathcal{M}_n$ to $\mathcal{M}_n$.

4. Let $\|\cdot\|$ denote the usual Euclidean norm on $\mathbb{R}^n$. Show that the map sending x to $\|x\|^2$ is differentiable everywhere. What is its derivative? Where is the map sending x to $\|x\|$ differentiable and what is its derivative?

5. Consider the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ for $x \neq 0$, and $f(0) = 0$. Show that f is differentiable except at 0, and that

$$Df(x)(h) = \frac{h}{\|x\|} - \frac{x(x \cdot h)}{\|x\|^3}.$$

Verify that $Df(x)(h)$ is orthogonal to x and explain geometrically why this is the case.

6. At which points of $\mathbb{R}^2$ is the function $f(x,y) = |x||y|$ differentiable? What about the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x,y) = xy/\sqrt{x^2+y^2}$ if $(x,y) \neq (0,0)$, $g(0,0) = 0$?

7. Show that the function $\det\colon \mathcal{M}_n \to \mathbb{R}$ is differentiable at the identity matrix I with $D\det(I)(H) = \operatorname{tr}(H)$. Deduce that det is differentiable at any invertible matrix A with $D\det(A)(H) = \det A \operatorname{tr}(A^{-1}H)$. Show further that det is twice differentiable at I and find $D^2\det(I)$ as a bilinear map.

8. Define $f\colon \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^2$. Show that f is continuously differentiable on the whole of $\mathcal{M}_n$. Deduce that there is a continuous square-root function on some neighbourhood of I; that is, show that there is an open ball $B_\varepsilon(I)$ for some $\varepsilon > 0$ and a continuous function $g\colon B_\varepsilon(I) \to \mathcal{M}_n$ such that $g(A)^2 = A$ for all $A \in B_\varepsilon(I)$. Is it possible to define a continuous square-root function on the whole of $\mathcal{M}_n$?

9. Let f be a real-valued function on a subset E of $\mathbb{R}^2$ such that that $f(\cdot, y)$ is continuous for each fixed $y \in E$ and $f(x, \cdot)$ is continuous for each fixed $x \in E$. Give an example to show that f need not be continuous on E. If additionally $f(\cdot, y)$ is Lipschitz for each $y \in E$ with Lipschitz constant independent of y and E has the property that $E \cap L$ is an open subset of L for every line L parallel to the y-axis, show that f is continuous on E. Deduce that if U is an open subset of $\mathbb{R}^2$, $f\colon U \to \mathbb{R}$, $D_1 f$ exists and is bounded on U and $f(x, \cdot)$ is continuous for each fixed $x \in U$, then f is continuous on U.

10. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $a \in \mathbb{R}^2$. If $D_1 f$ exists in some open ball around a and is continuous at a, and if $D_2 f$ exists at a, show that f is differentiable at a.

11. Let $C = \{(x,y) \in \mathbb{R}^2 : x^3 + y^3 - 3xy = 0\}$ and define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $F(x,y) = (x, x^3 + y^3 - 3xy)$. Show that F is locally C^1-invertible around each point of C except $(0,0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$; that is, show that if $(x_0, y_0) \in C \backslash \{(0,0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$ then there are open sets U containing (x_0, y_0) and V containing $F(x_0, y_0) = (x_0, 0)$ such that F maps U bijectively to V with inverse a C^1 function. What is the derivative of the inverse function? Deduce that for each point $(x_0, y_0) \in C \backslash \{(0,0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$, there exists an open interval $I \subset \mathbb{R}$ containing x_0 and a C^1 function $g\colon I \to \mathbb{R}$ such that $C \cap V = \operatorname{graph} g$ ($\operatorname{graph} g = \{(x, g(x)) : x \in I\}$).

12. (i) Let E be a subset of $\mathbb{R}$. Show that E is path-connected if and only if E is an interval, i.e. E is of the form (a, b), $[a, b)$, $(a, b]$ or $[a, b]$ for some a, b with $-\infty \leq a \leq b \leq \infty$. [Hint: Let $b = \sup E$ and $a = \inf E$ (allowing $\pm\infty$). Use the intermediate value theorem to show that if E is path-connected, then any x with $a < x < b$ belongs to E.]

(ii) Let U be a non-empty open subset of $\mathbb{R}^n$. Show that U is path-connected $\iff$ whenever $U = U_1 \cup U_2$ for disjoint open subsets U_1, U_2 of $\mathbb{R}^n$, either U_1 or U_2 is empty. [Hint: For the direction $\Rightarrow$, use the theorem that says that a function with zero derivative on a path-connected open set must be constant; for $\Leftarrow$, show first that the relation $x \sim y \iff$ there exists a continuous map $\gamma : [0, 1] \to U$ with $\gamma(0) = x$, $\gamma(1) = y$ is an equivalence relation on U with each equivalence class (called a path component) an open subset.]

13*. For $a, b \in \mathbb{R}^n$ and a continuous map $\gamma\colon [0, 1] \to \mathbb{R}^n$ with $\gamma(0) = a$, $\gamma(1) = b$, define the length $\ell(\gamma)$ of γ to be $\ell(\gamma) = \sup \sum_{j=1}^{N} \|\gamma(t_j) - \gamma(t_{j-1})\|$ where the sup is taken over all finite partitions $0 = t_0 < t_1 < \ldots < t_N = 1$.

(i) Give an example for which $\ell(\gamma) = \infty$. If γ is continuously differentiable on $[0, 1]$, show that $\ell(\gamma) < \infty$ and that in fact $\ell(\gamma) = \int_0^1 \|\gamma'(t)\|\, dt$.

(ii) For a path-connected subset E of $\mathbb{R}^n$ and $a, b \in E$, define $d(a, b) = \inf \ell(\gamma)$, where the inf is taken over all continuous $\gamma\colon [0, 1] \to E$ with $\gamma(0) = a$, $\gamma(1) = b$. Show, for any $a, b, c \in E$, that $d(a, b) \geq 0$ with equality iff $a = b$, that $d(a, b) = d(b, a)$ and that $d(a, b) \leq d(a, c) + d(c, b)$.

14*. Let U be a path-connected open subset of $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}^m$ be differentiable on U with $\|Df(x)\| \leq M$ for some constant M and all $x \in U$. Does it follow that $\|f(b) - f(a)\| \leq M\|b - a\|$ for every $a, b \in U$? Does it follow that $\|f(b) - f(a)\| \leq M d(a, b)$ for every $a, b \in U$, where d is as in Q13(ii) with $E = U$?

15*. (i) Let f be a real-valued C^2 function on an open subset U of $\mathbb{R}^2$. If f has a local maximum at a point $a \in U$ (meaning that there is $\rho > 0$ such that $B_\rho(a) \subset U$ and $f(x) \leq f(a)$ for every $x \in B_\rho(a)$), show that $Df(a) = 0$ and that the matrix $H = (D_{ij} f(a))$ is negative semi-definite (i.e. has non-positive eigenvalues).

(ii) Let U be a bounded open subset of $\mathbb{R}^2$ and let $f\colon \overline{U} \to \mathbb{R}$ be continuous on $\overline{U}$ (the closure of U) and C^2 in U. If f satisfies the partial differential inequality $\Delta f + aD_1 f + bD_2 f + cf \geq 0$ in U where Δ is the Laplace's operator defined by $\Delta f = D_{11} f + D_{22} f$, and a, b, c are real-valued functions on U with $c < 0$ on U, and if f is positive somewhere in $\overline{U}$, show that

$$\sup_{\overline{U}} f = \sup_{\partial U} f$$

where $\partial U = \overline{U} \setminus U$ is the boundary of U. Deduce that if a, b, c are as above, $\varphi\colon \partial U \to \mathbb{R}$ is a given continuous function, then for any $g\colon \mathbb{R}^2 \to \mathbb{R}$ there is at most one continuous function f on $\overline{U}$ that is C^2 in U and solves the boundary value problem $\Delta f + aD_1 f + bD_2 f + cf = g$ in U, $f = \varphi$ on ∂U.