ANALYSIS II—EXAMPLES 3 Mich. 2014

The questions marked with * are intended as additional; attempt them if you have time after the first 10 questions. Please email comments, corrections to: `n.wickramasekera@dpmms.cam.ac.uk`.

1. Quickies:

(i) Is the set $(1,2]$ an open subset of the metric space $\mathbb{R}$ with the metric $d(x,y)=|x-y|$? Is it closed? What if we replace the metric space $\mathbb{R}$ with the space $[0,2]$, the space$(1,3)$ or the space $(1,2]$, in each case with the metric d?

(ii) Let X be a set equipped with the discrete metric, and Y any metric space. Describe all open subsets of X, closed subsets of X, sequentially compact subsets of X, Cauchy sequences in X, continuous functions $X \to Y$ and continuous functions $Y \to X$.

(iii) If (X,d) is a metric space and A is a non-empty subset of X, show that the distance from $x \in X$ to A defined by $\rho(x) = \inf_{y\in A} d(x,y)$ is a Lipschitz function on X with Lipschitz constant 1.

(iv) If every closed, bounded subset of a metric space X is sequentially compact, must X be complete?

(v) If every closed proper subset of a metric space X with the induced metric is complete, must X be complete?

(vi) If (x_n), (y_n) are Cauchy sequences in a metric space (X,d), show that $(d(x_n,y_n))$ is convergent (in $\mathbb{R}$).

2. For each of the following sets X, determine whether or not the given function d defines a metric on X. In each case where the function does define a metric, describe the open ball $B_\varepsilon(x)$ for $x \in X$ and $\varepsilon > 0$ small.

(i) $X = \mathbb{R}^n$; $d(x,y) = \min\{|x_1-y_1|, |x_2-y_2|, \ldots, |x_n-y_n|\}$.

(ii) $X = \mathbb{Z}$; $d(x,x)=0$, and, for $x \neq y$, $d(x,y) = 2^n$ where $x - y = 2^n a$ with n a non-negative integer and a an odd integer.

(iii) X is the set of functions from $\mathbb{N}$ to $\mathbb{N}$; $d(f,f)=0$, and, for $f \neq g$, $d(f,g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

(iv) $X = \mathbb{C}$; $d(z,w) = |z-w|$ if z and w lie on the same line through the origin, $d(z,w) = |z|+|w|$ otherwise.

3. Let (X,d) be a metric space.

(a) Show that the union of any collection of open subsets of X must be open (regardless of whether the collection is finite, countable or uncountable), and that the intersection of any collection of closed subsets must be closed.

(b) Let E be a subset of X. Show that there is a unique largest open subset E^o of X contained in E, i.e. a unique open subset E^o of X such that that $E^o \subseteq E$ and if G is any open subset of X with $G \subseteq E$ then $G \subseteq E^o$. E^o is called the *interior* of E in X. Show also that there is a unique smallest closed subset $\overline{E}$ of X containing E, i.e. a unique closed subset $\overline{E}$ of X with $E \subseteq \overline{E}$ and if F is any closed subset of X with $E \subseteq F$ then $\overline{E} \subseteq F$. $\overline{E}$ is called the *closure* of E in X.

(c) Show that

$$E^o = \{x \in X \,:\, B_\epsilon(x) \subset E \quad \text{for} \quad \text{some} \quad \epsilon > 0\}$$

and that

$$\overline{E} = \{x \in X \,:\, x_n \to x \quad \text{for} \quad \text{some} \quad \text{sequence} \quad (x_n) \quad \text{in} \quad E\}.$$

4. Let V be a normed space, $x \in V$ and $r > 0$. Prove that the closure of the open ball $B_r(x)$ is the closed ball $D_r(x) = \{y \in V : \|x-y\| \leq r\}$. Give an example to show that, in a general metric space (X,d), the closure of the open ball $B_r(x)$ need not be the closed ball $D_r(x) = \{y \in X : d(x,y) \leq r\}$.

5. In lectures we proved that if E is a closed and bounded subset of $\mathbb{R}^n$, then any continuous function on E has bounded image. Prove the converse: if E is a subset of $\mathbb{R}^n$ and if every continuous function $f : E \to \mathbb{R}$ has bounded image, then E is closed and bounded.

6. Each of the following properties/notions makes sense for an arbitrary metric spaces X. Which are topological (i.e. dependent only on the collection of open subsets of X and not, in particular, on the metric on X generating the open subsets)? Justify your answers.

(i) boundedness of a subset of X.

(ii) closed-ness of a subset of X.

(iii) notion that a subset of X is closed *and* bounded.

(iv) total boundedness of X; that is, the property that for every $\epsilon > 0$, there is a finite set $F \subset X$ such that the union of open balls with centres in F and radius ϵ is X.

(v) completeness of X.

(vi) total boundedness *and* completeness of X.

7. Use the Contraction Mapping Theorem to show that the equation $\cos x = x$ has a unique real solution. Find this solution to some reasonable accuracy using a calculator (remember to work in radians!), and justify the claimed accuracy of your approximation.

8. Let $I = [0, R]$ be an interval and let $C(I)$ be the space of continuous functions on I. Show that, for any $\alpha \in \mathbb{R}$, we may define a norm by $\|f\|_\alpha = \sup_{x \in I} |f(x)e^{-\alpha x}|$, and that the norm $\|\cdot\|_\alpha$ is Lipschitz equivalent to the uniform norm $\|f\| = \sup_{x \in I} |f(x)|$.
Now suppose that $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, and Lipschitz in the second variable. Consider the map T from $C(I)$ to itself sending f to $y_0 + \int_0^x \phi(t, f(t))dt$. Give an example to show that T need not be a contraction under the uniform norm. Show, however, that T is a contraction under the norm $\|\cdot\|_\alpha$ for some α, and hence deduce that the differential equation $f'(x) = \phi(x, f(x))$ has a unique solution on I satisfying $f(0) = y_0$.

9. Let (X, d) be a non-empty complete metric space. Suppose $f\colon X \to X$ is a contraction and $g\colon X \to X$ is a function which commutes with f, i.e. such that $f(g(x)) = g(f(x))$ for all $x \in X$. Show that g has a fixed point. Must this fixed point be unique?

10. Give an example of a non-empty complete metric space (X, d) and a function $f\colon X \to X$ satisfying $d(f(x), f(y)) < d(x, y)$ for all x, $y \in X$ with $x \neq y$, but such that f has no fixed point. Suppose now that X is a non-empty closed bounded subset of $\mathbb{R}^n$ with the Euclidean metric. Show that in this case f must have a fixed point. If $g\colon X \to X$ satisfies $d(g(x), g(y)) \leq d(x, y)$ for all x, $y \in X$, must g have a fixed point?

11.* Show that it is not possible to obtain, starting from an arbitrary set $X \subseteq \mathbb{R}^n$ and repeatedly applying the operations $(\cdot)^o$ (interior) and $\overline{(\cdot)}$ (closure), more than seven distinct sets (including X itself). Give an example in $\mathbb{R}$ where seven sets are obtained.

12.* Let (X, d) be a non-empty complete metric space and let $f\colon X \to X$ be a function such that for each positive integer n we have
(i) if $d(x, y) < n + 1$ then $d(f(x), f(y)) < n$; and
(ii) if $d(x, y) < 1/n$ then $d(f(x), f(y)) < 1/(n + 1)$.
Must f have a fixed point?

13.* Let K be a closed bounded subset of $\mathbb{R}$ and $p \in K$. Construct a metric d on $K_1 = K \setminus \{p\}$ such that (K_1, d) is complete and the topology generated by d on K_1 is the same as the topology generated by the Euclidean metric on K_1.

14.* Let $(V, \|\cdot\|)$ be a normed space. Show that V is complete if and only if every absolutely convergent sequence in V is convergent, i.e. if and only if $\sum_{n=1}^\infty x_n$ is convergent whenever $\sum_{n=1}^\infty \|x_n\|$ is convergent. [One direction of this was Q8(a) on sheet 2; for the other (which should have been part (b) of that question!), show first that if (x_n) is Cauchy, then there is a subsequence (x_{n_j}) such that $\sum_j \|x_{n_{j+1}} - x_{n_j}\| < \infty$.]

15.* For each $n \in \mathbb{N}$, let $f_n \in C([0, 1])$ be such that $f_n(0) = 0$ and f_n is continuously differentiable on $[0, 1]$ with $\int_0^1 |f_n'|^2 < n^{-2}$. Show that there exists a subsequence (f_{n_j}) converging uniformly to zero on $[0, 1]$.