ANALYSIS II—EXAMPLES 2 Mich. 2014

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1. Let $(x^{(m)})$ and $(y^{(m)})$ be sequences in $\mathbb{R}^n$ converging to x and y respectively. Show that $x^{(m)} \cdot y^{(m)}$ converges to $x \cdot y$. Deduce that if $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are continuous at $x \in \mathbb{R}^n$, then so is the pointwise scalar product function $f \cdot g : \mathbb{R}^n \to \mathbb{R}$.

2. (a) Show that $\|f\|_1 = \int_0^1 |f|$ defines a norm on the vector space $C\big([0,1]\big)$. Is it Lipschitz equivalent to the uniform norm? Is $C\big([0,1]\big)$ with norm $\|\cdot\|_1$ complete?

(b) Let $R\big([0,1]\big)$ denote the vector space of all (Riemann) integrable functions on $[0,1]$. Does $\|f\|_1 = \int_0^1 |f|$ define a norm on $R\big([0,1]\big)$? If so, is $R\big([0,1]\big)$ complete with this norm? What if we replace $\|\cdot\|_1$ with $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$?

3. (a) Let $C^1([0,1])$ be the vector space of real continuous functions on $[0,1]$ with continuous first derivatives. Define functions $\alpha, \beta, \gamma, \delta : C^1([0,1]) \to \mathbb{R}$ by $\alpha(f) = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|$; $\beta(f) = \sup_{x\in[0,1]} (|f(x)| + |f'(x)|)$; $\gamma(f) = \sup_{x\in[0,1]} |f(x)|$; $\delta(f) = \sup_{x\in[0,1]} |f'(x)|$. Which of these define norms on $C^1([0,1])$? Out of those that define norms, which pairs are Lipschitz equivalent?

(b) Let $C^1_c([0,1])$ be the set of functions $f \in C^1([0,1])$ such that $f(x) = 0$ for x in some neighborhood of the end points 0 and 1. Verify that $C^1_c([0,1])$ is a vector space. How would your answers in (a) change if we replace $C^1([0,1])$ by $C^1_c([0,1])$?

4. Which of the following subsets of $\mathbb{R}^2$ with the Euclidean norm are open? Which are closed? (And why?)
(i) $\{(x,0) : 0 \le x \le 1\}$;
(ii) $\{(x,0) : 0 < x < 1\}$;
(iii) $\{(x,y) : y \ne 0\}$;
(iv) $\{(x,y) : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$;
(v) $\{(x,y) : y = nx \text{ for some } n \in \mathbb{N}\} \cup \{(x,y) : x = 0\}$;
(vi) $\{(x, f(x)) : x \in \mathbb{R}\}$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

5. Is the set $\{f : f(1/2) = 0\}$ closed in the space $C\big([0,1]\big)$ with the uniform norm? What about the set $\{f : \int_0^1 f = 0\}$? In each case, does the answer change if we replace the uniform norm with the norm $\|\cdot\|_1$ defined in Q2?

6. Which of the following functions f are continuous?
(i) The linear map $f\colon \ell^\infty \to \mathbb{R}$ defined by $f(x) = \sum_{n=1}^\infty x_n/n^2$;
(ii) The identity map from the space $C\big([0,1]\big)$ with the uniform norm $\|\cdot\|$ to the space $C\big([0,1]\big)$ with the norm $\|\cdot\|_1$ defined in Q2;
(iii) The identity map from $C\big([0,1]\big)$ with the norm $\|\cdot\|_1$ to $C\big([0,1]\big)$ with the uniform norm $\|\cdot\|$;
(iv) The linear map $f\colon \ell^0 \to \mathbb{R}$ defined by $f(x) = \sum_{i=1}^\infty x_i$, where ℓ^0 has norm $\|\cdot\|_\infty$. (ℓ^0 is the space of real sequences (x_k) such that $x_k = 0$ for all but a finite number of k.)

7. (a) Let ℓ^1 denote the set of real sequences (x_n) such that $\sum_{n=1}^\infty |x_n|$ is convergent. Show that, with addition and scalar multiplication defined pointwise, ℓ^1 is a vector space. Define $\|\cdot\|_1\colon \ell^1 \to \mathbb{R}$ by $\|x\|_1 = \sum_{n=1}^\infty |x_n|$. Show that $\|\cdot\|_1$ is a norm on ℓ^1, and that $(\ell^1, \|\cdot\|_1)$ is complete.

(b)* For $0 \le p < \infty$, let ℓ^p be the set of real sequences (x_n) such that $\sum_{n=1}^\infty |x_n|^p$ is convergent, and define $\|\cdot\|_p : \ell^p \to \mathbb{R}$ by $\|x\|_p = \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p}$. Generalise your results in (a) to the case $(\ell^p, \|\cdot\|_p)$ for any real p with $1 \le p < \infty$.

8. (a) Let $(V, \|\cdot\|)$ be a complete normed space and (x_n) a sequence in V such that $\sum_{n=1}^\infty \|x_n\|$ converges. Show that $\sum_{n=1}^\infty x_n$ converges.

9. Let V be a normed space in which every bounded sequence has a convergent subsequence. (a) Show that V must be complete. (b)* Show further that V must be finite-dimensional.
[Hint for (b): Start by showing that for every finite-dimensional subspace V_0 of V, there exists $x \in V$ with $\|x+y\| > \|x\|/2$ for each $y \in V_0$.]

10. Let $(x^{(n)})_{n\geq 1}$ be a bounded sequence in ℓ^∞. Show that there is a subsequence $(x^{(n_j)})_{j\geq 1}$ which converges in every coordinate; that is to say, the sequence $(x_i^{(n_j)})_{j\geq 1}$ of real numbers converges for each i. Why does this not show that every bounded sequence in ℓ^∞ has a convergent subsequence?

11. Is it possible to find uncountably many norms on $C([0,1])$ such that no two are Lipschitz equivalent?

12. (a) Let $(V, \|\cdot\|)$ be a complete normed space, and let W be a subspace of V. Show that $(W, \|\cdot\|)$ is complete if and only if W is closed in V.

(b) Which of the following vector spaces of functions, taken with the uniform norm, are complete?
(i) The space $C_b(\mathbb{R})$ of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$.
(ii) The space $C_0(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$.
(iii) The space $C_c(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $|x|$ sufficiently large.

13. We say that a family $\mathcal{F}$ of real functions on a closed, bounded interval $[a,b]$ is *equicontinuous* if for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $f \in \mathcal{F}$, $x, y \in [a,b]$ and $|x-y| < \delta$. If S is a subset of $C([a,b])$ such that every sequence in S has a subsequence converging uniformly to a function in S (i.e. if S is sequentially compact with respect to the uniform norm), show that S is closed, bounded and equicontinuous. (Note that this is the converse to the Arzela-Ascoli theorem which was mentioned in lecture).

14. * In lectures we proved that $\|f\|_p = \left(\int_0^1 |f|^p\right)^{1/p}$ defines a norm on $C([0,1])$ for $1 \leq p < \infty$. If $f_n(x) = \sin 2\pi n x$, show that the sequence (f_n) is bounded in $(C([0,1]), \|\cdot\|_2)$, but has no convergent subsequence. What if we take the norm $\|\cdot\|_p$ instead, with $2 < p < \infty$, or with $1 \leq p < 2$?