ANALYSIS II—EXAMPLES 1 Mich. 2014

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1. Which of the following sequences (f_n) of functions converge uniformly on the set X?
(a) $f_n(x) = x^n$ on $X = (0, 1)$; (b) $f_n(x) = x^n$ on $X = (0, \frac{1}{2})$; (c) $f_n(x) = xe^{-nx}$ on $X = [0, \infty)$;
(d) $f_n(x) = e^{-x^2} \sin(x/n)$ on $X = \mathbb{R}$.

2. Let (f_n) and (g_n) be sequences of real-valued functions on a subset of $\mathbb{R}$ converging uniformly to f and g respectively. Show that the pointwise sum $f_n + g_n$ converges uniformly to $f + g$. On the other hand, show that the pointwise product $f_n g_n$ need not converge uniformly to fg, but that if both f and g are bounded then $f_n g_n$ does converge uniformly to fg. What if f is bounded but g is not?

3. Let (f_n) be a sequence of bounded, real-valued functions on a subset of $\mathbb{R}$ converging uniformly to a function f. Show that f must be bounded. Give an example of a sequence (g_n) of bounded, real-valued functions on $[-1, 1]$ converging pointwise to a function g which is not bounded.

4. Let (f_n) be a sequence of real-valued continuous functions on a closed, bounded interval $[a, b]$, and suppose that f_n converges pointwise to a continuous function f. Show that if $f_n \to f$ uniformly and (x_m) is a sequence of points in $[a, b]$ with $x_m \to x$ then $f_n(x_n) \to f(x)$. On the other hand, show that if f_n does not converge uniformly to f then we can find a convergent sequence $x_m \to x$ in $[a, b]$ such that $f_n(x_n) \not\to f(x)$.

5. Let (f_n) be a sequence of real-valued functions on $[0, 1]$ converging uniformly to a function f.
(a) If $\mathcal{D}_n$ is the set of discontinuities of f_n and $\mathcal{D}$ is the set of discontinuities of f, show that $\mathcal{D} \subseteq \cup_{n=1}^{\infty} \cap_{j=n}^{\infty} \mathcal{D}_j$.
(b) Suppose that for some finite k, each f_n is discontinuous at most at k points. What can you say about the set of discontinuities of f?

6. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers.
(a) Define a sequence (f_n) of functions on $[-\pi, \pi]$ by $f_n(x) = \sum_{m=1}^{n} a_m \sin mx$. Show that each f_n is differentiable with $f_n'(x) = \sum_{m=1}^{n} m a_m \cos mx$.
(b) Show that $f(x) = \sum_{m=1}^{\infty} a_m \sin mx$ defines a continuous function on $[-\pi, \pi]$, but that the series $\sum_{m=1}^{\infty} m a_m \cos mx$ need not converge.

7. Show that, for any $x \in X = \mathbb{R} - \{1, 2, 3, \dots\}$, the series $\sum_{m=1}^{\infty} (x - m)^{-2}$ converges. Define $f \colon X \to \mathbb{R}$ by $f(x) = \sum_{m=1}^{\infty} (x - m)^{-2}$, and for $n = 1, 2, 3, \dots$, define $f_n \colon X \to \mathbb{R}$ by $f_n(x) = \sum_{m=1}^{n} (x - m)^{-2}$. Does the sequence (f_n) converge uniformly to f on X? Is f continuous?

8. Let a_n be real numbers such that $\sum_{n=0}^{\infty} a_n$ converges.
(a) Show that $\sum_{n=1}^{\infty} a_n x^n$ converges for $x \in (-1, 1)$. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, show that f is differentiable on $(-1, 1)$.
(b)* Show that f extends to $(-1, 1]$ as a continuous function with $f(1) = \sum_{n=0}^{\infty} a_n$. (Hint: start by showing that $f(x) = (1 - x) \sum_{n=0}^{\infty} s_n x^n$ for $|x| < 1$, where $s_n = \sum_{j=0}^{n} a_j$.) Show that, for each $r \in (-1, 1)$, the series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[r, 1]$. Must the one-sided derivative $f'(1)$ exist?

9. Is there a real power series with radius of convergence 1 that converges uniformly on $(-1, 1)$?

10. Which of the following functions $f \colon [0, \infty) \to \mathbb{R}$ are (a) uniformly continuous; (b) bounded?
(i) $f(x) = \sin x^2$; (ii) $f(x) = \inf\{|x - n^2| : n \in \mathbb{N}\}$; (iii) $f(x) = (\sin x^3)/(x + 1)$.

11. Show that if (f_n) is a sequence of uniformly continuous, real-valued functions on $\mathbb{R}$, and if $f_n \to f$ uniformly, then f is uniformly continuous. Give an example of a sequence of uniformly continuous, real-valued functions (f_n) on $\mathbb{R}$ such that f_n converges pointwise to a function f which is continuous but not uniformly continuous.

12. Suppose that $f\colon [0,\infty) \to \mathbb{R}$ is continuous, and that $f(x)$ tends to a (finite) limit as $x \to \infty$. Must f be uniformly continuous on $[0,\infty)$? Give a proof or counterexample as appropriate.

13. Let f be a differentiable, real-valued function on $\mathbb{R}$, and suppose that f' is bounded. Show that f is uniformly continuous. Let $g\colon [-1,1] \to \mathbb{R}$ be the function defined by $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $g(0) = 0$. Show that g is differentiable, but that its derivative is unbounded. Is g uniformly continuous?

14. Let f be a bounded real-valued Riemann integrable functions on $[0,1]$.

(a) Must there exist a sequence (f_n) of continuous functions on $[0,1]$ such that $f_n \to f$ uniformly on $[0,1]$?

(b)* Must there exist a sequence (f_n) of continuous functions on $[0,1]$ such that $\int_0^1 |f_n(x) - f(x)|\, dx \to 0$?

(c)* Must there exist a sequence (p_n) of polynomials such that $\int_0^1 |p_n(x) - f(x)|\, dx \to 0$?

15*. Define $\varphi(x) = |x|$ for $x \in [-1,1]$ and extend the definition of $\varphi(x)$ to all real x by requiring that $\varphi(x+2) = \varphi(x)$.

(i) Show that $|\varphi(s) - \varphi(t)| \leq |s-t|$ for all s and t.

(ii) Define $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$. Prove that f is well-defined and continuous.

(iii) Fix a real number x and positive integer m. Put $\delta_m = \pm\frac{1}{2}4^{-m}$, where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x+\delta_m)$. Prove that

$$\left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| \geq \frac{1}{2}(3^m + 1).$$

Conclude that f is not differentiable at x. Hence there exists a real continuous function on the real line which is nowhere differentiable.