Mich. 2013 **ANALYSIS II—EXAMPLES 3** PAR

1. Let (X, d) and (Y, e) be metric spaces, with X non-empty. Suppose that Y is bounded, i.e. there is some $M \in \mathbb{R}$ such that $e(x, y) \leq M$ for all x, $y \in Y$. Let Z be the set of functions from X to Y. Show that $D(f, g) = \sup_{x \in X} e(f(x), g(x))$ defines a metric on Z.

2. Let V be a normed space, $x \in V$ and $r > 0$. Prove that the closure of the open ball $B_r(x)$ is the closed ball $A_r(x) = \{y \in V : \|x - y\| \leq r\}$. Give an example to show that, in a general metric space (X, d), the closure of the open ball $B_r(x)$ need not be the closed ball $A_r(x) = \{y \in X : d(x, y) \leq r\}$.

3. (a) Show that $\|f\|_1 = \int_0^1 |f|$ defines a norm on the vector space $C\big([0, 1]\big)$. Is it Lipschitz equivalent to the uniform norm? Is $C\big([0, 1]\big)$ with norm $\| \cdot \|_1$ complete?

(b) Let $R\big([0, 1]\big)$ denote the vector space of all (Riemann) integrable functions on $[0, 1]$. Does $\|f\|_1 = \int_0^1 |f|$ define a norm on $R\big([0, 1]\big)$? If so, is $R\big([0, 1]\big)$ complete with this norm? What if we replace $\| \cdot \|_1$ with $\|f\|_\infty = \sup\{|f(x)| : x \in [0, 1]\}$?

4. Is the set $\{f : f(1/2) = 0\}$ closed in the space $C\big([0, 1]\big)$ with the uniform norm? What about the set $\{f : \int_0^1 f = 0\}$? In each case, does the answer change if we replace the uniform norm with the norm $\| \cdot \|_1$?

5. Which of the following functions f are continuous?
(i) The linear map $f \colon \ell^\infty \to \mathbb{R}$ defined by $f(x) = \sum_{n=1}^\infty x_n/n^2$;
(ii) The identity map from the space $C\big([0, 1]\big)$ with the uniform norm $\| \cdot \|$ to the space $C\big([0, 1]\big)$ with the norm $\| \cdot \|_1$ as defined in Q3;
(iii) The identity map from $C\big([0, 1]\big)$ with the norm $\| \cdot \|_1$ to $C\big([0, 1]\big)$ with the uniform norm $\| \cdot \|$;
(iv) The linear map $f \colon \ell^0 \to \mathbb{R}$ defined by $f(x) = \sum_{i=1}^\infty x_i$, where ℓ^0 has norm $\| \cdot \|_\infty$.

6. Let ℓ^1 denote the vector space of real sequences (x_n) such that $\sum_{n=1}^\infty x_n$ is absolutely convergent, with addition and scalar multiplication defined pointwise. Define $\| \cdot \|_1 \colon \ell^1 \to \mathbb{R}$ by $\|x\|_1 = \sum_{n=1}^\infty |x_n|$. Show that $\| \cdot \|_1$ is a norm, and that ℓ^1 endowed with this norm is complete.

7. (a) Let $(V, \| \cdot \|)$ be a complete normed space and (x_n) a sequence in V such that $\sum_{n=1}^\infty \|x_n\|$ converges. Show that $\sum_{n=1}^\infty x_n$ converges.

8. Let V be a normed space in which every bounded sequence has a convergent subsequence. Show that V must be complete. $^+$Show further that V must be finite-dimensional.

9. Let $(x^{(n)})_{n \geq 1}$ be a bounded sequence in ℓ^∞. Show that there is a subsequence $(x^{(n_j)})_{j \geq 1}$ which converges in every coordinate; that is to say, the sequence $(x_i^{(n_j)})_{j \geq 1}$ of real numbers converges for each i. Why does this not show that every bounded sequence in ℓ^∞ has a convergent subsequence?

10. Is it possible to find uncountably many norms on ℓ^0 such that no two are Lipschitz equivalent?

11. Let (X, d) be a non-empty complete metric space. Suppose $f \colon X \to X$ is a contraction and $g \colon X \to X$ is a function which commutes with f, i.e. such that $f(g(x)) = g(f(x))$ for all $x \in X$. Show that g has a fixed point. Must this fixed point be unique?

12. Give an example of a non-empty complete metric space (X, d) and a function $f \colon X \to X$ satisfying $d(f(x), f(y)) < d(x, y)$ for all x, $y \in X$ with $x \neq y$, but such that f has no fixed point. Suppose now that X is a non-empty closed bounded subset of $\mathbb{R}^n$ with the Euclidean metric. Show that in this case f must have a fixed point. If $g \colon X \to X$ satisfies $d(g(x), g(y)) \leq d(x, y)$ for all x, $y \in X$, must g have a fixed point?

13. Let (X, d) be a non-empty complete metric space and let $f \colon X \to X$ be a function such that for each positive integer n we have
(i) if $d(x, y) < n + 1$ then $d(f(x), f(y)) < n$; and
(ii) if $d(x, y) < 1/n$ then $d(f(x), f(y)) < 1/(n + 1)$.
Must f have a fixed point?