1. Let $(x^{(m)})$ and $(y^{(m)})$ be sequences in $\mathbb{R}^n$ converging to x and y respectively. Show that $x^{(m)} \cdot y^{(m)} \to x \cdot y$. Deduce that if $f\colon \mathbb{R}^n \to \mathbb{R}^p$ and $g\colon \mathbb{R}^n \to \mathbb{R}^p$ are continuous at $x \in \mathbb{R}^n$ then so is the pointwise scalar product function $f \cdot g\colon \mathbb{R}^n \to \mathbb{R}$.

2. Define $f\colon [0,\pi] \to \mathbb{C}$ by $f(x) = e^{ix}$. Calculate $\int_0^\pi f$.

3. Which of the following subsets of $\mathbb{R}^2$ are open? Which are closed? (And why?)
(i) $\{(x,0) : 0 \le x \le 1\}$;
(ii) $\{(x,0) : 0 < x < 1\}$;
(iii) $\{(x,y) : y \ne 0\}$;
(iv) $\{(x,y) : y = nx \text{ for some } n \in \mathbb{N}\} \cup \{(x,y) : x = 0\}$;
(v) $\{(x,y) : y = qx \text{ for some } q \in \mathbb{Q}\} \cup \{(x,y) : x = 0\}$;
(vi) $\{(x, f(x)) : x \in \mathbb{R}\}$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function.

4. Is the set $\{(x, x\sin\frac{1}{x}) : x \in (0,\infty)\} \cup \{(0,0)\}$ a path-connected subset of $\mathbb{R}^2$? What about the set $\{(x, \sin\frac{1}{x}) : x \in (0,\infty)\} \cup \{(0,0)\}$?

5. Let $K \subset \mathbb{R}^n$ and let $f\colon K \to \mathbb{R}^m$ be continuous on K. If K is closed, must $f(K)$ be closed? If K is bounded, must $f(K)$ be bounded?

6. Is the set $(1,2]$ an open subset of the metric space $\mathbb{R}$ with metric $d(x,y) = |x-y|$? Is it closed? What if we replace the metric space $\mathbb{R}$ by the metric space $[0,2]$, the metric space $(1,3)$ or the metric space $(1,2]$, in each case with metric $d(x,y) = |x-y|$?

7. For each of the following sets X, determine whether or not the given function d defines a metric on X. In each case where the function does define a metric, describe the open ball $B_\varepsilon(x)$ for $x \in X$ and $\varepsilon > 0$ small.
(i) $X = \mathbb{R}^n$; $d(x,y) = \min\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$.
(ii) $X = \mathbb{Z}$; $d(x,x) = 0$, and, for $x \ne y$, $d(x,y) = 2^n$ where $x - y = 2^n a$ with n a non-negative integer and a an odd integer.
(iii) X is the set of functions from $\mathbb{N}$ to $\mathbb{N}$; $d(f,f) = 0$, and, for $f \ne g$, $d(f,g) = 2^{-n}$ for the least n such that $f(n) \ne g(n)$.
(iv) $X = \mathbb{C}$; $d(z,w) = |z-w|$ if z and w lie on the same line through the origin, $d(z,w) = |z| + |w|$ otherwise.

8. Let d and d' denote the usual and discrete metrics respectively on $\mathbb{R}$. Show that all functions f from $\mathbb{R}$ with metric d' to $\mathbb{R}$ with metric d are continuous. What are the continuous functions from $\mathbb{R}$ with metric d to $\mathbb{R}$ with metric d'?

9. For X, $Y \subset \mathbb{R}^n$, define $X + Y = \{x + y : x \in X, y \in Y\}$. Give examples of closed sets X, $Y \subset \mathbb{R}^n$ for some n such that $X + Y$ is not closed. Show that it is not possible to find such an example with X bounded. If V, $W \subset \mathbb{R}^n$ are open, must $V + W$ be open?

10. (a) Show that the union of any collection of open subsets of $\mathbb{R}^n$ must be open (regardless of whether the collection be finite or infinite, countable or uncountable), and that the intersection of any collection of closed subsets of $\mathbb{R}^n$ must be closed.

(b) We define the *interior* of a set $X \subset \mathbb{R}^n$ to be the largest open set X° contained in X, and the *closure* of $X \subset \mathbb{R}^n$ to be the smallest closed set $\bar{X}$ containing X. Why does the result of (a) tell us that these definitions make sense?

(c) Show that

$$X^\circ = \{x \in X : B_\varepsilon(x) \subset X \text{ for some } \varepsilon > 0\}$$

and that

$$\bar{X} = \{x \in \mathbb{R}^n : x^{(m)} \to x \text{ for some sequence } (x^{(m)}) \text{ in } X\}.$$

11. Starting from an arbitrary set $X \subset \mathbb{R}^n$ and repeatedly applying the operations $(\cdot)^\circ$ and $\overline{(\cdot)}$, show that it is not possible to obtain more than seven distinct sets (including X itself). Give an example in $\mathbb{R}$ where seven distinct sets are obtained.

12. Does there exist a continuous surjection $f\colon \mathbb{R} \to \mathbb{R}^2$? Does there exist a continuous injection $f\colon \mathbb{R}^2 \to \mathbb{R}$? In each case, what happens if we replace $\mathbb{R}^2$ with the metric space ℓ^∞?

$^+$13. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function under which the image of any path-connected set is path-connected and the image of any closed bounded set is closed and bounded. Show that f must be continuous.