Mich. 2013 **ANALYSIS II—EXAMPLES 1** PAR

1. Which of the following sequences (f_n) of functions converge uniformly on the set X?
(a) $f_n(x) = x^n$ on $X = (0, 1)$; (b) $f_n(x) = x^n$ on $X = (0, \frac{1}{2})$; (c) $f_n(x) = xe^{-nx}$ on $X = [0, \infty)$;
(d) $f_n(x) = e^{-x^2} \sin(x/n)$ on $X = \mathbb{R}$.

2. Construct a sequence (f_n) of *continuous* real-valued functions on $[-1, 1]$ converging pointwise to the zero function but with $\int_{-1}^{1} f_n \not\to 0$. $^+$Is it possible to find such a sequence with $|f_n(x)| \le 1$ for all n and for all x?

3. Let (f_n) and (g_n) be sequences of real-valued functions on a subset X of $\mathbb{R}$ converging uniformly to f and g respectively. Show that the pointwise sum $f_n + g_n$ converges uniformly to $f + g$. On the other hand, show that the pointwise product $f_n g_n$ need not converge uniformly to fg, but that if both f and g are bounded then $f_n g_n$ does converge uniformly to fg. What if f is bounded but g is not?

4. Let (f_n) be a sequence of real-valued continuous functions on a closed, bounded interval $[a, b]$, and suppose that f_n converges pointwise to a continuous function f. Show that if $f_n \to f$ uniformly and (x_m) is a sequence of points in $[a, b]$ with $x_m \to x$ then $f_n(x_n) \to f(x)$. On the other hand, show that if f_n does not converge uniformly to f then we can find a convergent sequence $x_m \to x$ in $[a, b]$ such that $f_n(x_n) \not\to f(x)$.

5. Which of the following functions $f\colon [0, \infty) \to \mathbb{R}$ are (a) uniformly continuous; (b) bounded?
(i)$f(x) = \sin x^2$; (ii) $f(x) = \inf\{|x - n^2| : n \in \mathbb{N}\}$; (iii) $f(x) = (\sin x^3)/(x + 1)$.

6. Show that if (f_n) is a sequence of uniformly continuous, real-valued functions on $\mathbb{R}$, and if $f_n \to f$ uniformly, then f is uniformly continuous. Give an example of a sequence of uniformly continuous, real-valued functions (f_n) on $\mathbb{R}$ such that f_n converges pointwise to a function f which is continuous but not uniformly continuous.

7. Suppose that $f\colon [0, \infty) \to \mathbb{R}$ is continuous, and that $f(x)$ tends to a (finite) limit as $x \to \infty$. Must f be uniformly continuous on $[0, \infty)$? Give a proof or counterexample as appropriate.

8. Is there a real power series with radius of convergence 1 that converges uniformly on $(-1, 1)$?

9. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers.
(a) Define a sequence (f_n) of functions on $[-\pi, \pi]$ by $f_n(x) = \sum_{m=1}^{n} a_m \sin mx$. Show that each f_n is differentiable with $f_n'(x) = \sum_{m=1}^{n} m a_m \cos mx$.
(b) Show that $f(x) = \sum_{m=1}^{\infty} a_m \sin mx$ defines a continuous function on $[-\pi, \pi]$, but that the series $\sum_{m=1}^{\infty} m a_m \cos mx$ need not converge.

10. Show that, for any $x \in X = \mathbb{R} - \{1, 2, 3, \dots\}$, the series $\sum_{m=1}^{\infty} (x - m)^{-2}$ converges. Define $f\colon X \to \mathbb{R}$ by $f(x) = \sum_{m=1}^{\infty} (x - m)^{-2}$, and for $n = 1, 2, 3, \dots$, define $f_n\colon X \to \mathbb{R}$ by $f_n(x) = \sum_{m=1}^{n} (x - m)^{-2}$. Does the sequence (f_n) converge uniformly to f on X? Is f continuous?

11. Let f be a differentiable, real-valued function on $\mathbb{R}$, and suppose that f' is bounded. Show that f is uniformly continuous. Let $g\colon [-1, 1] \to \mathbb{R}$ be the function defined by $g(x) = x^2 \sin(1/x^2)$ for $x \ne 0$ and $g(0) = 0$. Show that g is differentiable, but that its derivative is unbounded. Is g uniformly continuous?

12. Construct a function $f\colon [0, 1] \to \mathbb{R}$ which is not the pointwise limit of any sequence of continuous functions.

13. Let (f_n) be a sequence of continuous, real-valued functions on $[0, 1]$ converging pointwise to a function f. Prove that there is some subinterval $[a, b]$ of $[0, 1]$ with $a < b$ on which f is bounded.