Mich. 2012 **ANALYSIS II—EXAMPLES 4** PAR

1. Use the contraction mapping theorem to show that the equation $\cos x = x$ has a unique real solution. Find this solution to some reasonable accuracy using an electronic pocket calculator, and justify the claimed accuracy of your approximation.

2. Let (X, d) be a non-empty complete metric space. Suppose $f\colon X \to X$ is a contraction and $g\colon X \to X$ is a function which commutes with f, i.e. such that $f(g(x)) = g(f(x))$ for all $x \in X$. Show that g has a fixed point. Must this fixed point be unique?

3. Let $I = [0, R]$ be an interval and let $C(I)$ be the space of continuous functions on I. Show that, for any $\alpha \in \mathbb{R}$, we may define a norm by $\|f\|_\alpha = \sup_{x\in I} |f(x)e^{-\alpha x}|$, and that the norm $\|\cdot\|_\alpha$ is Lipschitz equivalent to the uniform norm $\|f\| = \sup_{x\in I} |f(x)|$.

Now suppose that $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, and Lipschitz in the second variable. Consider the map $T\colon C(I) \to C(I)$ defined by $T(f)(x) = y_0 + \int_0^x \phi(t, f(t))dt$. Give an example to show that T need not be a contraction under the uniform norm. Show, however, that T is a contraction under the norm $\|\cdot\|_\alpha$ for some α, and hence deduce that the differential equation $f'(x) = \phi(x, f(x))$ has a unique solution on I satisfying $f(0) = y_0$.

4. Define $f\colon \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^4$. Show that f is differentiable at every $A \in \mathcal{M}_n$, and find $Df|_A$ as a linear map. Show further that f is twice-differentiable at every $A \in \mathcal{M}_n$ and find $D^2f|_A$ as a bilinear map from $\mathcal{M}_n \times \mathcal{M}_n$ to $\mathcal{M}_n$.

5. Let $\|\cdot\|$ denote the usual Euclidean norm on $\mathbb{R}^n$. Show that the map sending x to $\|x\|^2$ is differentiable everywhere. What is its derivative? Where is the map sending x to $\|x\|$ differentiable and what is its derivative?

6. We work in $\mathbb{R}^3$ with the Euclidean norm. Consider the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ for $x \neq 0$, and $f(0) = 0$. Show that f is differentiable except at 0, and that

$$Df|_x(h) = \frac{h}{\|x\|} - \frac{x(x\cdot h)}{\|x\|^3}.$$

Verify that $Df|_x(h)$ is orthogonal to x and explain geometrically why this is the case.

7. At which points is the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = |x||y|$ differentiable? What about the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x, y) = \begin{cases} xy/\sqrt{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}?$$

8. Show that the function $\det\colon \mathcal{M}_n \to \mathbb{R}$ is differentiable at the identity matrix I with $D\det|_I(H) = \mathrm{tr}(H)$. Deduce that det is differentiable at any invertible matrix A with $D\det|_A(H) = \det A\,\mathrm{tr}(A^{-1}H)$. Show further that det is twice differentiable at I and find $D^2\det|_I$ as a bilinear map.

9. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $(a,b) \in \mathbb{R}^2$.

(a) Suppose that $D_1 f$ exists and is continuous in some open ball around (a,b), and that $D_2 f$ exists at (a,b). Show that f is differentiable at (a,b).

(b) Suppose instead that $D_1 f$ exists and is bounded on some open ball around (a,b), and that for fixed x the function $y \mapsto f(x,y)$ is continuous. Show that f is continuous at (a,b).

10. Define $f\colon \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^2$. Show that f is continuously differentiable on the whole of $\mathcal{M}_n$. Deduce that there is a continuous square-root function on some neighbourhood of I; that is, show that there is an open ball $B_\varepsilon(I)$ for some $\varepsilon > 0$ and a continuous function $g\colon B_\varepsilon(I) \to \mathcal{M}_n$ such that $g(A)^2 = A$ for all $A \in B_\varepsilon(I)$. Is it possible to define a continuous square-root function on the whole of $\mathcal{M}_n$?

11. Give an example of a non-empty complete metric space (X,d) and a function $f\colon X \to X$ satisfying $d(f(x), f(y)) < d(x,y)$ for all x, $y \in X$ with $x \neq y$, but such that f has no fixed point. Suppose now that X is a non-empty closed bounded subset of $\mathbb{R}^n$ with the Euclidean metric. Show that in this case f must have a fixed point. If $g\colon X \to X$ satisfies $d(g(x), g(y)) \leq d(x,y)$ for all x, $y \in X$, must g have a fixed point?

12. Let (X,d) be a non-empty complete metric space and let $f\colon X \to X$ be a function such that for each positive integer n we have
(i) if $d(x,y) < n+1$ then $d(f(x), f(y)) < n$; and
(ii) if $d(x,y) < 1/n$ then $d(f(x), f(y)) < 1/(n+1)$.
Must f have a fixed point?

13. Let (X,d) be a non-empty complete metric space, let $f\colon X \to X$ be a continuous function, and let $K \in [0,1)$.

(a) Suppose we assume that for all x, $y \in X$ we have either $d(f(x), f(y)) \leq K d(x,y)$ or $d(f(f(x)), f(f(y))) \leq K d(x,y)$. Show that f has a fixed point.

$^+$(b) Suppose instead we assume only that for all x, $y \in X$ at least one of the three distances $d(f(x), f(y))$, $d(f(f(x)), f(f(y)))$ and $d(f(f(f(x))), f(f(f(y))))$ is less than or equal to $K d(x,y)$. Must f have a fixed point?