Mich. 2012 **ANALYSIS II—EXAMPLES 3** PAR

1. Is the set $(1,2]$ an open subset of the metric space $\mathbb{R}$ with metric $d(x,y)=|x-y|$? Is it closed? What if we replace the metric space $\mathbb{R}$ by the metric space $[0,2]$, the metric space $(1,3)$ or the metric space $(1,2]$, in each case with metric $d(x,y)=|x-y|$?

2. For each of the following sets X, determine whether or not the given function d defines a metric on X. In each case where the function does define a metric, describe the open ball $B_\varepsilon(x)$ for $x\in X$ and $\varepsilon>0$ small.
 (i) $X=\mathbb{R}^n$; $d(x,y)=\min\{|x_1-y_1|,|x_2-y_2|,\dots,|x_n-y_n|\}$.
 (ii) $X=\mathbb{Z}$; $d(x,x)=0$, and, for $x\neq y$, $d(x,y)=2^n$ where $x-y=2^n a$ with n a non-negative integer and a an odd integer.
 (iii) X is the set of functions from $\mathbb{N}$ to $\mathbb{N}$; $d(f,f)=0$, and, for $f\neq g$, $d(f,g)=2^{-n}$ for the least n such that $f(n)\neq g(n)$.
 (iv) $X=\mathbb{C}$; $d(z,w)=|z-w|$ if z and w lie on the same line through the origin, $d(z,w)=|z|+|w|$ otherwise.

3. Let d and d' denote the usual and discrete metrics respectively on $\mathbb{R}$. Show that all functions f from $\mathbb{R}$ with metric d' to $\mathbb{R}$ with metric d are continuous. What are the continuous functions from $\mathbb{R}$ with metric d to $\mathbb{R}$ with metric d'?

4. Let V be a normed space, $x\in V$ and $r>0$. Prove that the closure of the open ball $B_r(x)$ is the closed ball $A_r(x)=\{y\in V:\|x-y\|\le r\}$. Give an example to show that, in a general metric space (X,d), the closure of the open ball $B_r(x)$ need not be the closed ball $A_r(x)=\{y\in X:d(x,y)\le r\}$.

5. (a) Show that $\|f\|_1=\int_0^1|f|$ defines a norm on the vector space $C\big([0,1]\big)$. Is it Lipschitz equivalent to the uniform norm? Is $C\big([0,1]\big)$ with norm $\|\cdot\|_1$ complete?

(b) Let $R\big([0,1]\big)$ denote the vector space of all (Riemann) integrable functions on $[0,1]$. Does $\|f\|_1=\int_0^1|f|$ define a norm on $R\big([0,1]\big)$?

6. Is the set $\{f: f(1/2)=0\}$ closed in the space $C\big([0,1]\big)$ with the uniform norm? What about the set $\{f:\int_0^1 f=0\}$? In each case, does the answer change if we replace the uniform norm with the norm $\|\cdot\|_1$?

7. Which of the following functions f are continuous?
 (i) The linear map $f\colon\ell^\infty\to\mathbb{R}$ defined by $f(x)=\sum_{n=1}^\infty x_n/n^2$;
 (ii) The identity map from the space $C\big([0,1]\big)$ with the uniform norm $\|\cdot\|$ to the space $C\big([0,1]\big)$ with the norm $\|\cdot\|_1$ as defined in Q3;
 (iii) The identity map from $C\big([0,1]\big)$ with the norm $\|\cdot\|_1$ to $C\big([0,1]\big)$ with the uniform norm $\|\cdot\|$;
 (iv) The linear map $f\colon\ell^0\to\mathbb{R}$ defined by $f(x)=\sum_{i=1}^\infty x_i$.

8. Let ℓ^1 denote the vector space of real sequences (x_n) such that $\sum_{n=1}^\infty x_n$ is absolutely convergent, with addition and scalar multiplication defined pointwise. Define $\|\cdot\|_1\colon\ell^1\to\mathbb{R}$ by $\|x\|_1=\sum_{n=1}^\infty|x_n|$. Show that $\|\cdot\|_1$ is a norm, and that ℓ^1 endowed with this norm is complete.

9. Let $(V,\|\cdot\|)$ be a complete normed space and (x_n) a sequence in V such that $\sum_{n=1}^\infty\|x_n\|$ converges. Show that $\sum_{n=1}^\infty x_n$ converges.

10. Let V be a normed space in which every bounded sequence has a convergent subsequence. Show that V must be complete. $^+$Show further that V must be finite-dimensional.

11. Let $(x^{(n)})_{n\ge1}$ be a bounded sequence in ℓ^∞. Show that there is a subsequence $(x^{(n_j)})_{j\ge1}$ which converges in every coordinate; that is to say, the sequence $(x_i^{(n_j)})_{j\ge1}$ of real numbers converges for each i. Why does this not show that every bounded sequence in ℓ^∞ has a convergent subsequence?

12. Is it possible to find uncountably many norms on ℓ^0 such that no two are Lipschitz equivalent?

13. Does there exist a continuous surjection $f\colon\mathbb{R}\to\ell^\infty$?