1. Let f, $g\colon \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Show that the function $f+g\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous.

2. Let $f\colon \mathbb{R}^n \to \mathbb{R}^p$ be continuous at the point $x \in \mathbb{R}^n$, and let $g\colon \mathbb{R}^p \to \mathbb{R}^q$ be continuous at the point $f(x) \in \mathbb{R}^p$. Show that the composition $g \circ f\colon \mathbb{R}^n \to \mathbb{R}^q$ is continuous at x.

3. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function and let $x \in \mathbb{R}^n$. Suppose that for any sequence $(x^{(k)})$ in $\mathbb{R}^n$ with $x^{(k)} \to x$ we have $f(x^{(k)}) \to f(x)$. Show that f must be continuous at x.

4. Show that a sequence $(x^{(n)})$ in $\mathbb{R}^m$ is Cauchy if and only if the sequences $(x_i^{(n)})$ are Cauchy for each i with $1 \leqslant i \leqslant m$.

5. Which of the following subsets of $\mathbb{R}^2$ are open? Which are closed? (And why?)
(i) $\{(x,0) : 0 \leq x \leq 1\}$;
(ii) $\{(x,0) : 0 < x < 1\}$;
(iii) $\{(x,y) : y \neq 0\}$;
(iv) $\{(x,y) : y = nx \text{ for some } n \in \mathbb{N}\} \cup \{(x,y) : x = 0\}$;
(v) $\{(x,y) : y = qx \text{ for some } q \in \mathbb{Q}\} \cup \{(x,y) : x = 0\}$;
(vi) $\{(x,f(x)) : x \in \mathbb{R}\}$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function.

6. Is the set $\{(x, x\sin\frac{1}{x}) : x \in (0,\infty)\} \cup \{(0,0)\}$ a path-connected subset of $\mathbb{R}^2$? What about the set $\{(x, \sin\frac{1}{x}) : x \in (0,\infty)\} \cup \{(0,0)\}$?

7. Let $(x^{(n)})$ be a sequence in $\mathbb{R}^m$ such that $\sum_{n=1}^{\infty} \|x^{(n)}\|$ converges. Show that $\sum_{n=1}^{\infty} x^{(n)}$ converges.

8. Let $K \subset \mathbb{R}^n$ and let $f\colon K \to \mathbb{R}^m$ be continuous on K. If K is closed, must $f(K)$ be closed? If K is bounded, must $f(K)$ be bounded?

9. For X, $Y \subset \mathbb{R}^n$, define $X + Y = \{x + y : x \in X, y \in Y\}$. Give examples of closed sets X, $Y \subset \mathbb{R}^n$ for some n such that $X+Y$ is not closed. Show that it is not possible to find such an example with X bounded. If V, $W \subset \mathbb{R}^n$ are open, must $V+W$ be open?

10. (a) Show that the union of any collection of open subsets of $\mathbb{R}^n$ must be open (regardless of whether the collection be finite or infinite, countable or uncountable), and that the intersection of any collection of closed subsets of $\mathbb{R}^n$ must be closed.
(b) We define the *interior* of a set $X \subset \mathbb{R}^n$ to be the largest open set X° contained in X, and the *closure* of $X \subset \mathbb{R}^n$ to be the smallest closed set $\bar{X}$ containing X. Why does the result of (a) tell us that these definitions make sense?
(c) Show that

$$X^\circ = \{x \in X : B_\varepsilon(x) \subset X \text{ for some } \varepsilon > 0\}$$

and that

$$\bar{X} = \{x \in \mathbb{R}^n : x^{(m)} \to x \text{ for some sequence } (x^{(m)}) \text{ in } X\}.$$

11. Starting from an arbitrary set $X \subset \mathbb{R}^n$ and repeatedly applying the operations $(\cdot)^\circ$ and $\bar{(\cdot)}$, show that it is not possible to obtain more than seven distinct sets (including X itself). Give an example in $\mathbb{R}$ where seven distinct sets are obtained.

12. Does there exist a continuous surjection $f\colon \mathbb{R} \to \mathbb{R}^2$? Does there exist a continuous injection $f\colon \mathbb{R}^2 \to \mathbb{R}$?

$^+$13. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function under which the image of any path-connected set is path-connected and the image of any closed bounded set is closed and bounded. Show that f must be continuous.