

## ANALYSIS II (Michaelmas 2011): EXAMPLES 1

The questions are not equally difficult. Those marked with \* are intended as ‘additional’, to be attempted if you wish to take things further. Comments, corrections are welcome at any time and may be sent to [a.j.scholl@dpmmms.cam.ac.uk](mailto:a.j.scholl@dpmmms.cam.ac.uk).

1. Which of the following sequences of functions converge uniformly on  $X$ ?

- (a)  $f_n(x) = x^n$  on  $X = (0, \frac{1}{2})$ ;
- (b)  $f_n(x) = \sin(n^2x)/\log n$  on  $X = \mathbb{R}$ ;
- (c)  $f_n(x) = x^n$  on  $X = (0, 1)$ ;
- (d)  $f_n(x) = x^n - x^{2n}$  on  $X = [0, 1]$ ;
- (e)  $f_n(x) = xe^{-nx}$  on  $X = [0, \infty)$ ;
- (f)  $f_n(x) = e^{-x^2} \sin(x/n)$  on  $X = \mathbb{R}$ .

2. Suppose that  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous. Show that the sequence  $(x^n f(x))$  is uniformly convergent on  $[0, 1]$  if and only if  $f(1) = 0$ .

3. Let  $f$  and  $g$  be uniformly continuous real-valued functions on a set  $E \subseteq \mathbb{R}$ . Show that the pointwise sum  $f + g$  is uniformly continuous on  $E$ , and so is  $\lambda f$  for each real constant  $\lambda$ . Give an example showing that the (pointwise) product  $fg$  need not be uniformly continuous on  $E$ . Is it possible to find such an example with  $f$  bounded?

4. Let  $(f_n)$  be a sequence of continuous real-valued functions on a closed, bounded interval  $[a, b]$ , and suppose that  $f_n$  converges pointwise to a continuous function  $f$ .

Show that if  $f_n \rightarrow f$  uniformly on  $[a, b]$  and  $(x_m)$  is a sequence of points in  $[a, b]$  with  $x_m \rightarrow x$ , then  $f_n(x_m) \rightarrow f(x)$ . [Careful — this is not quite as easy as it looks!]

On the other hand, show that if  $f_n$  does **not** converge uniformly to  $f$ , then we can find a convergent sequence  $x_m \rightarrow x$  in  $[a, b]$  such that  $f_n(x_m)$  does not converge to  $f(x)$ .

[Hint: Bolzano–Weierstrass.]

5. Which of the following functions  $f$  on  $[0, \infty)$  are (a) uniformly continuous, (b) bounded?

- (i)  $f(x) = \sin x^2$ ;
- (ii)  $f(x) = \inf\{|x - n^2| : n \in \mathbb{N}\}$ ;
- (iii)  $f(x) = (\sin x^3)/(x + 1)$ .

6. Suppose that  $f : [0, \infty) \rightarrow \mathbb{R}$  is continuous and that  $f(x)$  tends to a (finite) limit as  $x \rightarrow \infty$ . Is  $f$  necessarily uniformly continuous on  $[0, \infty)$ ? Give a proof or a counter-example as appropriate.

7. Show that if  $(f_n)$  is a sequence of uniformly continuous functions on  $\mathbb{R}$ , and  $f_n \rightarrow f$  uniformly on  $\mathbb{R}$ , then  $f$  is uniformly continuous. Give an example of a sequence of uniformly continuous functions  $f_n$  on  $\mathbb{R}$ , such that  $f_n$  converges pointwise to a continuous function  $f$ , but  $f$  is not uniformly continuous.

[Hint for the last part: choose the limit function  $f$  first.]

8. Let  $f_n(x) = n^\alpha x^n(1 - x)$ , where  $\alpha$  is a real constant.

- (i) For which values of  $\alpha$  does  $f_n(x) \rightarrow 0$  pointwise on  $[0, 1]$ ?
- (ii) For which values of  $\alpha$  does  $f_n(x) \rightarrow 0$  uniformly on  $[0, 1]$ ?
- (iii) For which values of  $\alpha$  does  $\int_0^1 f_n(x) dx \rightarrow 0$ ?
- (iv) For which values of  $\alpha$  does  $f'_n(x) \rightarrow 0$  pointwise on  $[0, 1]$ ?
- (v) For which values of  $\alpha$  does  $f'_n(x) \rightarrow 0$  uniformly on  $[0, 1]$ ?

**9.** Consider the sequence of functions  $f_n : \mathbb{R} \setminus \mathbb{Z} \rightarrow \mathbb{R}$  defined by  $f_n(x) = \sum_{m=-n}^n (x-m)^{-2}$ . Show that  $f_n$  converges pointwise on  $\mathbb{R} \setminus \mathbb{Z}$  to a function  $f$ . Show that  $f_n$  does not converge uniformly on  $\mathbb{R} \setminus \mathbb{Z}$ . Why can we nevertheless conclude that the limit function  $f$  is continuous, and indeed differentiable, on  $\mathbb{R} \setminus \mathbb{Z}$ ?

**10.** Let  $f$  be a differentiable, real-valued function on a (bounded or unbounded) interval  $E \subseteq \mathbb{R}$ , and suppose that  $f'$  is bounded on  $E$ . Show that  $f$  is uniformly continuous on  $E$ .

Let  $g : [-1, 1] \rightarrow \mathbb{R}$  be the function defined by  $g(x) = x^2 \sin(1/x^2)$ , for  $x \neq 0$  and  $g(0) = 0$ . Show that  $g$  is differentiable, but its derivative is unbounded. Is  $g$  uniformly continuous on  $[-1, 1]$ ?

**11.** Suppose that a function  $f$  has a continuous derivative on  $(a, b) \subseteq \mathbb{R}$  and

$$f_n(x) = n\left(f\left(x + \frac{1}{n}\right) - f(x)\right).$$

Show that  $f_n$  converges uniformly to  $f'$  on each interval  $[\alpha, \beta] \subset (a, b)$ .

**12.** Let  $\sum_{n=1}^{\infty} a_n$  be an absolutely convergent series of real numbers. Define a sequence  $(f_n)$  of functions on  $[-\pi, \pi]$  by  $f_n(x) = \sum_{m=1}^n a_m \sin mx$  and show that each  $f_n$  is differentiable with  $f'_n(x) = \sum_{m=1}^n m a_m \cos mx$ .

Show further that  $f(x) = \sum_{m=1}^{\infty} a_m \sin mx$  defines a continuous function on  $[-\pi, \pi]$ , but that the series  $\sum_{m=1}^{\infty} m a_m \cos mx$  need not converge.

**13.\*** Let  $f$  be a bounded function defined on a set  $E \subseteq \mathbb{R}$ , and for each positive integer  $n$  let  $g_n$  be a function defined on  $E$  by

$$g_n(x) = \sup\{|f(y) - f(x)| : y \in E, |y - x| < 1/n\}.$$

Show that  $f$  is uniformly continuous on  $E$  if and only if  $g_n \rightarrow 0$  uniformly on  $E$  as  $n \rightarrow \infty$ .

**14.\*** (Dini's theorem) Let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be a sequence of continuous functions converging pointwise to a continuous function  $f : [0, 1] \rightarrow \mathbb{R}$ . Suppose that  $f_n(x)$  is a decreasing sequence  $f_n(x) \geq f_{n+1}(x)$  for each  $x \in [0, 1]$ . Show that  $f_n \rightarrow f$  uniformly on  $[0, 1]$ .

[If you have done Metric and Topological Spaces then you may prefer to find a topological proof.]

**15.\*** (Abel's test) Let  $a_n$  and  $b_n$  be real-valued functions on  $E \subseteq \mathbb{R}$ . Suppose that  $\sum_{n=0}^{\infty} a_n(x)$  is uniformly convergent on  $E$ . Suppose further that the  $b_n(x)$  are uniformly bounded on  $E$  (this means there is a constant  $K$  with  $|b_n(x)| \leq K$  for all  $n$  and all  $x \in E$ ), and that  $b_n(x) \geq b_{n+1}(x)$  for all  $n$  and all  $x \in E$ . Show that the sum  $\sum_{n=0}^{\infty} a_n(x)b_n(x)$  is uniformly convergent on  $E$ .

[Hint: show first that  $\sum_{k=n}^m a_k b_k = \sum_{k=n}^{m-1} (b_k - b_{k+1}) A_k + b_m A_m - b_n A_{n-1}$ , where  $A_n = \sum_{k=0}^n a_k$ .]

Deduce that if  $a_n$  are real constants and  $\sum_{n=0}^{\infty} a_n$  is convergent, then  $\sum_{n=0}^{\infty} a_n x^n$  is uniformly convergent on  $[0, 1]$ . (But note that  $\sum_{n=0}^{\infty} a_n x^n$  need not be convergent at  $x = -1$ ; you almost certainly know a counterexample!)

**16.\*** Define  $\varphi(x) = |x|$  for  $x \in [-1, 1]$  and extend the definition of  $\varphi(x)$  to all real  $x$  by requiring that

$$\varphi(x+2) = \varphi(x).$$

(i) Show that  $|\varphi(s) - \varphi(t)| \leq |s - t|$  for all  $s$  and  $t$ .

(ii) Define  $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$ . Prove that  $f$  is well-defined and continuous.

(iii) Fix a real number  $x$  and positive integer  $m$ . Put  $\delta_m = \pm \frac{1}{2} 4^{-m}$ , where the sign is so chosen that no integer lies between  $4^m x$  and  $4^m(x + \delta_m)$ . Prove that

$$\left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| \geq \frac{1}{2} (3^m + 1).$$

Conclude that  $f$  is not differentiable at  $x$ . Hence there exists a real continuous function on the real line which is nowhere differentiable.