ANALYSIS II (Michaelmas 2010): EXAMPLES 1

The questions are not equally difficult. Those marked with * are intended as 'additional', to be attempted if you wish to take things further. Comments, corrections are welcome at any time and may be sent to `a.j.scholl@dpmms.cam.ac.uk`.

1. Which of the following sequences of functions converge uniformly on X?

(a) $f_n(x) = x^n$ on $X = (0, \frac{1}{2})$;
(b) $f_n(x) = \sin(n^2 x)/\log n$ on $X = \mathbb{R}$;
(c) $f_n(x) = x^n$ on $X = (0, 1)$;
(d) $f_n(x) = x^n - x^{2n}$ on $X = [0, 1]$;
(e) $f_n(x) = xe^{-nx}$ on $X = [0, \infty)$;
(f) $f_n(x) = e^{-x^2} \sin(x/n)$ on $X = \mathbb{R}$.

2. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous. Show that the sequence $(x^n f(x))$ is uniformly convergent on $[0, 1]$ if and only if $f(1) = 0$.

3. Let f and g be uniformly continuous real-valued functions on a set $E \subseteq \mathbb{R}$. Show that the pointwise sum $f + g$ is uniformly continuous on E, and so is λf for each real constant λ. Give an example showing that the (pointwise) product fg need not be uniformly continuous on E. Is it possible to find such an example with f bounded?

4. Let (f_n) be a sequence of continuous real-valued functions on a closed, bounded interval $[a, b]$, and suppose that f_n converges pointwise to a continuous function f.

Show that if $f_n \to f$ uniformly on $[a, b]$ and (x_m) is a sequence of points in $[a, b]$ with $x_m \to x$, then $f_n(x_n) \to f(x)$. [Careful — this is not quite as easy as it looks!]

On the other hand, show that if f_n does **not** converge uniformly to f, then we can find a convergent sequence $x_m \to x$ in $[a, b]$ such that $f_n(x_n)$ does not converge to $f(x)$.
[Hint: Bolzano–Weierstrass.]

5. Which of the following functions f on $[0, \infty)$ are (a) uniformly continuous, (b) bounded?

(i) $f(x) = \sin x^2$;
(ii) $f(x) = \inf\{|x - n^2| : n \in \mathbb{N}\}$;
(iii) $f(x) = (\sin x^3)/(x + 1)$.

6. Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous and that $f(x)$ tends to a (finite) limit as $x \to \infty$. Is f necessarily uniformly continuous on $[0, \infty)$? Give a proof or a counter-example as appropriate.

7. Show that if (f_n) is a sequence of uniformly continuous functions on $\mathbb{R}$, and $f_n \to f$ uniformly on $\mathbb{R}$, then f is uniformly continuous. Give an example of a sequence of uniformly continuous functions f_n on $\mathbb{R}$, such that f_n converges pointwise to a continuous function f, but f is not uniformly continuous.
[Hint for the last part: choose the limit function f first.]

8. Let $f_n(x) = n^\alpha x^n (1 - x)$, where α is a real constant.

(i) For which values of α does $f_n(x) \to 0$ pointwise on $[0, 1]$?
(ii) For which values of α does $f_n(x) \to 0$ uniformly on $[0, 1]$?
(iii) For which values of α does $\int_0^1 f_n(x)dx \to 0$?
(iv) For which values of α does $f_n'(x) \to 0$ pointwise on $[0, 1]$?
(v) For which values of α does $f_n'(x) \to 0$ uniformly on $[0, 1]$?

9. Consider the sequence of functions $f_n : \mathbb{R}\setminus\mathbb{Z} \to \mathbb{R}$ defined by $f_n(x) = \sum_{m=-n}^{n} (x-m)^{-2}$. Show that f_n converges pointwise on $\mathbb{R} \setminus \mathbb{Z}$ to a function f. Show that f_n does not converge uniformly on $\mathbb{R} \setminus \mathbb{Z}$. Why can we nevertheless conclude that the limit function f is continuous, and indeed differentiable, on $\mathbb{R} \setminus \mathbb{Z}$?

10. Let f be a differentiable, real-valued function on a (bounded or unbounded) interval $E \subseteq \mathbb{R}$, and suppose that f' is bounded on E. Show that f is uniformly continuous on E.

Let $g : [-1, 1] \to \mathbb{R}$ be the function defined by $g(x) = x^2 \sin(1/x^2)$, for $x \neq 0$ and $g(0) = 0$. Show that g is differentiable, but its derivative is unbounded. Is g uniformly continuous on $[-1, 1]$?

11. Suppose that a function f has a continuous derivative on $(a, b) \subseteq \mathbb{R}$ and

$$f_n(x) = n\big(f(x + \frac{1}{n}) - f(x)\big).$$

Show that f_n converges uniformly to f' on each interval $[\alpha, \beta] \subset (a, b)$.

12. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers. Define a sequence (f_n) of functions on $[-\pi, \pi]$ by $f_n(x) = \sum_{m=1}^{n} a_m \sin mx$ and show that each f_n is differentiable with $f_n'(x) = \sum_{m=1}^{n} m a_m \cos mx$.

Show further that $f(x) = \sum_{m=1}^{\infty} a_m \sin mx$ defines a continuous function on $[-\pi, \pi]$, but that the series $\sum_{m=1}^{\infty} m a_m \cos mx$ need not converge.

13.* Let f be a bounded function defined on a set $E \subseteq \mathbb{R}$, and for each positive integer n let g_n be a function defined on E by

$$g_n(x) = \sup\big\{|f(y) - f(x)| : y \in E,\ |y - x| < 1/n\big\}\ .$$

Show that f is uniformly continuous on E if and only if $g_n \to 0$ uniformly on E as $n \to \infty$.

14.* (Dini's theorem) Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuous functions converging pointwise to a continuous function $f : [0, 1] \to \mathbb{R}$. Suppose that $f_n(x)$ is a decreasing sequence $f_n(x) \geq f_{n+1}(x)$ for each $x \in [0, 1]$. Show that $f_n \to f$ uniformly on $[0, 1]$.
[If you have done Metric and Topological Spaces then you may prefer to find a topological proof.]

15.* (Abel's test) Let a_n and b_n be real-valued functions on $E \subseteq \mathbb{R}$. Suppose that $\sum_{n=0}^{\infty} a_n(x)$ is uniformly convergent on E. Suppose further that the $b_n(x)$ are uniformly bounded on E (this means there is a constant K with $|b_n(x)| \leq K$ for all n and all $x \in E$), and that $b_n(x) \geq b_{n+1}(x)$ for all n and all $x \in E$. Show that the sum $\sum_{n=0}^{\infty} a_n(x) b_n(x)$ is uniformly convergent on E.
[Hint: show first that $\sum_{k=n}^{m} a_k b_k = \sum_{k=n}^{m-1} (b_k - b_{k+1}) A_k + b_m A_m - b_n A_{n-1}$, where $A_n = \sum_{k=0}^{n} a_k$.]

Deduce that if a_n are real constants and $\sum_{n=0}^{\infty} a_n$ is convergent, then $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0, 1]$. (But note that $\sum_{n=0}^{\infty} a_n x^n$ need not be convergent at $x = -1$; you almost certainly know a counterexample!)

16.* Define $\varphi(x) = |x|$ for $x \in [-1, 1]$ and extend the definition of $\varphi(x)$ to all real x by requiring that

$$\varphi(x + 2) = \varphi(x).$$

(i) Show that $|\varphi(s) - \varphi(t)| \leq |s - t|$ for all s and t.

(ii) Define $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x)$. Prove that f is well-defined and continuous.

(iii) Fix a real number x and positive integer m. Put $\delta_m = \pm\frac{1}{2} 4^{-m}$, where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x + \delta_m)$. Prove that

$$\left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| \geq \frac{1}{2}(3^m + 1).$$

Conclude that f is not differentiable at x. Hence there exists a real continuous function on the real line which is nowhere differentiable.