ANALYSIS II (Michaelmas 2009): EXAMPLES 2

The questions are not equally difficult and the 'additional' ones are marked with *. Unless stated otherwise, the norm on $\mathbb{R}^n$ may be taken to be the Euclidean norm $\|x\|_2 = \sqrt{\sum_{i=1}^n x^2}$, and the spaces ℓ_0 and ℓ_∞ may be assumed to have the sup-norm $\|x\|_\infty = \sup_i |x_i|$. ($\ell_0$ denotes the space of real sequences $(x_n)_{n=1}^\infty$ such that all but finitely many x_n are zero.) Comments, corrections are welcome at any time and may be sent to `a.g.kovalev@dpmms.cam.ac.uk`.

1. Let $(x^{(m)})$ and $(y^{(m)})$ be sequences in $\mathbb{R}^n$ converging to x and y respectively. Show that $x^{(m)} \cdot y^{(m)}$ converges to $x \cdot y$. Deduce that if $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are continuous at $x \in \mathbb{R}^n$, then so is the pointwise scalar product function $f \cdot g : \mathbb{R}^n \to \mathbb{R}$.

2. Show that $\|x\|_1 = \sum_{i=1}^n |x_i|$ defines a norm on $\mathbb{R}^n$. Show directly that it is Lipschitz equivalent to the Euclidean norm.

3. (a) Show that $\|f\|_1 = \int_0^1 |f(x)|dx$ defines a norm on the space $C[0,1]$. Is it Lipschitz equivalent to the uniform norm?

(b) Let $R[0,1]$ denote the vector space of all integrable functions on $[0,1]$. Does $\|f\| = \int_0^1 |f(x)|dx$ define a norm on $R[0,1]$?

4. Which of the following subsets of $\mathbb{R}^2$ are open? Which are closed? (And why?)
(i) $\{(x,0) : 0 \le x \le 1\}$;
(ii) $\{(x,0) : 0 < x < 1\}$;
(iii) $\{(x,y) : y \neq 0\}$;
(iv) $\{(x,y) : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$;
(v) $\{(x,y) : y = nx \text{ for some } n \in \mathbb{N}\} \cup \{(x,y) : x = 0\}$;
(vi) $\{(x, f(x)) : x \in \mathbb{R}\}$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

5. Is the set $\{f : f(1/2) = 0\}$ closed in the space $C[0,1]$ with the uniform norm? What about the set $\{f : \int_0^1 f(x)dx = 0\}$? In each case, does the answer change if we replace the uniform norm with the norm $\|\cdot\|_1$ defined in Question 3?

6. Which of the following functions f are continuous?
(i) The linear map $f : \ell_\infty \to \mathbb{R}$ defined by $f(x) = \sum_{n=1}^\infty x_n/n^2$.
(ii) The identity map from the space $C[0,1]$ with the uniform norm to the space $C[0,1]$ with the norm $\|\cdot\|_1$ defined in Question 3.
(iii) The identity map from $C[0,1]$ with the norm $\|\cdot\|_1$ to $C[0,1]$ with the uniform norm.
(iv) The linear map $f : \ell_0 \to \mathbb{R}$ defined by $f(x) = \sum_{i=1}^\infty x_i$.

7. If A and B are subsets of $\mathbb{R}^n$, we write $A + B$ for the set $\{a + b : a \in A, b \in B\}$. Show that if A and B are both closed and one of them is bounded then $A + B$ is closed. Give an example in $\mathbb{R}^1$ to show that the boundedness condition cannot be omitted. If A and B are both open, is $A + B$ necessarily open? Justify your answer.

8. (a) Show that the space ℓ_∞ is complete. Show also that $c_0 = \{x \in \ell_\infty : x_n \to 0\}$, the vector subspace of ℓ_∞ consisting of all sequences converging to 0, is complete.

(b) Is the space $R[0,1]$ of integrable functions on $[0,1]$, equipped with the uniform norm, complete?

9. Let $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that $\|x\|' = \|x\| + \|\alpha x\|$ defines a norm on $\mathbb{R}^n$. Using the fact that all norms on a finite-dimensional space are Lipschitz equivalent, deduce that α is continuous.

10.* Which of the following vector spaces of functions, considered with the uniform norm, are complete? (Justify your answer.)

(i) The space $C_b(\mathbb{R})$ of bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$.

(ii) The space $C_0(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$.

(iii) The space $C_c(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $|x|$ sufficiently large.

11. In lectures we proved that if E is a closed and bounded set in $\mathbb{R}^n$, then any continuous function defined on E has bounded image. Prove the converse: if every continuous real-valued function on $E \subseteq \mathbb{R}^n$ is bounded, then E is closed and bounded.

12. Let $(x^{(m)})_{m\geq 1}$ be a bounded sequence in ℓ_∞. Show that there is a subsequence $(x^{(m_j)})_{j\geq 1}$ which converges in every coordinate; that is to say, the sequence $(x_i^{(m_j)})_{j\geq 1}$ of real numbers converges for each i. Why does this not show that every bounded sequence in ℓ_∞ has a convergent subsequence?

13. Show that $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ defines a norm on ℓ_0 and that this norm is not Lipschitz equivalent to the uniform norm $\|\cdot\|$. Find a third norm on ℓ_0 which is neither Lipschitz equivalent to $\|\cdot\|_1$, nor to $\|\cdot\|$. Is it possible to find uncountably many norms on ℓ_0 such that no two are Lipschitz equivalent?

14. Let V be a normed space in which every bounded sequence has a convergent subsequence. (a) Show that V must be complete. (b)* Show further that V must be finite-dimensional.

[Hint for (b): Show first that for every finite-dimensional subspace V_0 of V there exists an $x \in V$ with $\|x + y\| > \|x\|/2$ for each $y \in V_0$.]

15.* Recall from the lectures the normed space ℓ_2. The Hilbert cube is the subset of ℓ_2 consisting of all the sequences $(x_n)_{n=1}^{\infty}$ such that for each n, $|x_n| \leq 1/n$. Show that the Hilbert cube is closed in ℓ_2, and that it has the Bolzano–Weierstrass property, that is, any sequence in the Hilbert cube has a convergent subsequence. (So the Hilbert cube is *compact*.)