Mich. 2008 **ANALYSIS II—EXAMPLES 4** PAR

1. Show that the operator norm on the space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is indeed a norm.

2. Define $f: \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^4$. Show that f is differentiable at every $A \in \mathcal{M}_n$, and find $Df|_A$ as a linear map. Show further that f is twice-differentiable at every $A \in \mathcal{M}_n$ and find $D^2 f|_A$ as a bilinear map from $\mathcal{M}_n \times \mathcal{M}_n$ to $\mathcal{M}_n$.

3. Let $\|\cdot\|$ denote the usual Euclidean norm on $\mathbb{R}^n$. Show that the map sending x to $\|x\|^2$ is differentiable everywhere. What is its derivative? Where is the map sending x to $\|x\|$ differentiable and what is its derivative?

4. We work in $\mathbb{R}^3$ with the Euclidean norm. Consider the map $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ for $x \neq 0$, and $f(0) = 0$. Show that f is differentiable except at 0, and that

$$Df|_x(h) = \frac{h}{\|x\|} - \frac{x(x \cdot h)}{\|x\|^3}.$$

Verify that $Df|_x(h)$ is orthogonal to x and explain geometrically why this is the case.

5. At which points are the following functions $f: \mathbb{R}^2 \to \mathbb{R}$ differentiable?

(i) $f(x,y) = \begin{cases} x/y & y \neq 0 \\ 0 & y = 0 \end{cases}$;

(ii) $f(x,y) = |x||y|$;

(iii) $f(x,y) = xy|x-y|$;

(iv) $f(x,y) = \begin{cases} xy/\sqrt{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$;

(v) $f(x,y) = \begin{cases} xy\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

6. Let $\mathcal{N}_n \subset \mathcal{M}_n$ be the set of invertible $n \times n$ matrices. Show that $\mathcal{N}_n$ is an open subset of $\mathcal{M}_n$.

Define $f: \mathcal{N}_n \to \mathcal{N}_n$ by $f(A) = A^{-1}$. Show that f is differentiable at the identity matrix I, and that $Df|_I(H) = -H$.

Let $A \in \mathcal{N}_n$. By writing $(A+H)^{-1} = A^{-1}(I + HA^{-1})^{-1}$, or otherwise, show that f is differentiable at A. What is $Df|_A$?

Show further that f is twice-differentiable at I, and find $D^2 f|_I$ as a bilinear map.

7. Show that $\det: \mathcal{M}_n \to \mathbb{R}$ is differentiable at the identity matrix I with $D\det|_I(H) = \operatorname{tr}(H)$. Deduce that det is differentiable at any invertible matrix A with $D\det|_A(H) = \det A \operatorname{tr}(A^{-1}H)$. Show further that det is twice differentiable at I and find $D^2\det|_I$ as a bilinear map.

8. (a) Define $f: \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^4$. Find the Taylor series of $f(A+H)$ about A.

(b) Define $g: \mathcal{N}_n \to \mathcal{N}_n$ by $g(A) = A^{-1}$. Find the Taylor series of $g(I+H)$ about I.

9. Let $f: \mathbb{R}^2 \to \mathbb{R}$ and $(a,b) \in \mathbb{R}^2$.

(a) Suppose that $D_1 f$ exists and is continuous in some open ball around (a,b), and that $D_2 f$ exists at (a,b). Show that f is differentiable at (a,b).

(b) Suppose instead that $D_1 f$ exists and is bounded on some open ball around (a,b), and that for fixed x the function $y \mapsto f(x,y)$ is continuous. Show that f is continuous at (a,b).

10. Define $f\colon \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^2$. Show that f is continuously differentiable on the whole of $\mathcal{M}_n$. Deduce that there is a continuous square-root function on some neighbourhood of I; that is, show that there is an open ball $B_\varepsilon(I)$ for some $\varepsilon > 0$ and a continuous function $g\colon B_\varepsilon(I) \to \mathcal{M}_n$ such that $g(A)^2 = A$ for all $A \in B_\varepsilon(I)$.

Is it possible to define a square-root function on the whole of $\mathcal{M}_n$? What about a cube-root function?

11. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x,y) = (x, x^3 + y^3 - 3xy)$ and let $C = \{(x,y) \in \mathbb{R}^2 : x^3 + y^3 - 3xy = 0\}$. Show that f is locally invertible around each point of C except $(0,0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$; that is, show that if $(x_0, y_0) \in C \backslash \{(0,0), (2^{\frac{2}{3}}, 2^{\frac{1}{3}})\}$ then there are open sets U containing (x_0, y_0) and V containing $f(x_0, y_0)$ such that f maps U bijectively to V. What is the derivative of the local inverse function? Deduce that for each point $(x_0, y_0) \in C$ other than $(0,0)$ and $(2^{\frac{2}{3}}, 2^{\frac{1}{3}})$ there exist open intervals I containing x_0 and J containing y_0 such that for each $x \in I$ there is a unique $y \in J$ with $(x,y) \in C$.