1. (a) Is the set $(1, 2]$ an open subset of the metric space $\mathbb{R}$ with metric $d(x, y) = |x - y|$? Is it closed?

(b) Is the set $(1, 2]$ an open subset of the metric space $[0, 2]$ with metric $d(x, y) = |x - y|$? Is it closed?

(c) Is the set $(1, 2]$ an open subset of the metric space $(1, 3)$ with metric $d(x, y) = |x - y|$? Is it closed?

(d) Is the set $(1, 2]$ an open subset of the metric space $(1, 2] \cup (3, 4]$ with metric $d(x, y) = |x - y|$? Is it closed?

2. For each of the following sets X, determine whether or not the given function d defines a metric on X. In each case where the function does define a metric, describe the open ball $B_\varepsilon(x)$ for each $x \in X$ and $\varepsilon > 0$ small.

(i) $X = \mathbb{R}^n$; $d(x, y) = \min\{|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|\}$.

(ii) $X = \mathbb{Z}$; $d(x, x) = 0$ and for $x \neq y$, $d(x, y) = 2^n$ where $x - y = 2^n a$ with n a non-negative integer and a an odd integer.

(iii) X is the set of functions from $\mathbb{N}$ to $\mathbb{N}$; $d(f, f) = 0$, and for $f \neq g$, $d(f, g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

(iv) $X = \mathbb{C}$; $d(z, z) = 0$, and for $z \neq w$, $d(z, w) = |z| + |w|$.

(v) $X = \mathbb{C}$; $d(z, w) = |z - w|$ if z and w lie on the same straight line through the origin, $d(z, w) = |z| + |w|$ otherwise.

3. Let d and d' denote the usual and discrete metrics respectively on $\mathbb{R}$. Show that all functions f from $\mathbb{R}$ with metric d' to $\mathbb{R}$ with metric d are continuous. What are the continuous functions from $\mathbb{R}$ with metric d to $\mathbb{R}$ with metric d'?

4. (a) Show that the intersection of an arbitrary collection of closed subsets of a metric space must be closed.

(b) We define the *closure* of a subset Y of a metric space X to be the smallest closed set $\bar{Y}$ containing Y. Why does the result of (a) tell us that this definition makes sense?

(c) Show that

$$\bar{Y} = \{x \in X : x_n \to x \text{ for some sequence } (x_n) \text{ in } Y\}.$$

5. Let V be a normed space, $x \in V$ and $r > 0$. Prove that the closure of the open ball $B_r(x)$ is the closed ball $A_r(x) = \{y \in V : \|x - y\| \leq r\}$. Give an example to show that, in a general metric space (X, d), the closure of the open ball $B_r(x)$ need not be the closed ball $A_r(x) = \{y \in X : d(x, y) \leq r\}$.

6. Show that the equation $\cos x = x$ has a unique real solution. Find this solution to some reasonable accuracy using an electronic pocket calculator, and justify the claimed accuracy of your approximation.

7. Let $I = [0, R]$ be an interval and let $C(I)$ be the space of continuous functions on I. Show that, for any $\alpha \in \mathbb{R}$, we may define a norm by $\|f\|_\alpha = \sup_{x \in I} |f(x)e^{-\alpha x}|$, and that the norm $\|\cdot\|_\alpha$ is Lipschitz equivalent to the uniform norm $\|f\| = \sup_{x \in I} |f(x)|$.

Now suppose that $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, and Lipschitz in the second variable. Consider the map T from $C(I)$ to itself sending f to $y_0 + \int_0^x \phi(t, f(t))dt$. Give an example to show that T need not be a contraction under the uniform norm. Show, however, that T is a contraction under the norm $\|\cdot\|_\alpha$ for some α, and hence deduce that the differential equation $f'(x) = \phi(x, f(x))$ has a unique solution on I satisfying $f(0) = y_0$.

8. Let (X, d) be a non-empty complete metric space. Suppose $f\colon X \to X$ is a contraction and $g\colon X \to X$ is a function which commutes with f, i.e. such that $f(g(x)) = g(f(x))$ for all $x \in X$. Show that g has a fixed point. Must this fixed point be unique?

9. Give an example of a non-empty complete metric space (X, d) and a function $f: X \to X$ satisfying $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, but such that f has no fixed point. Suppose now that X is a non-empty closed bounded subset of $\mathbb{R}^n$ with the Euclidean metric. Show that in this case f must have a fixed point. If $g: X \to X$ satisfies $d(g(x), g(y)) \leq d(x, y)$ for all $x, y \in X$, must g have a fixed point?

10. Let (X, d) be a non-empty complete metric space and let $f: X \to X$ be a function such that for each positive integer n we have
(i) if $d(x, y) < n + 1$ then $d(f(x), f(y)) < n$; and
(ii) if $d(x, y) < 1/n$ then $d(f(x), f(y)) < 1/(n + 1)$.
Must f have a fixed point?

11. Let (X, d) be a non-empty complete metric space, let $f: X \to X$ be a continuous function, and let $K \in [0, 1)$.

(a) Suppose we assume that for all $x, y \in X$ we have either $d(f(x), f(y)) \leq Kd(x, y)$ or $d(f(f(x)), f(f(y))) \leq Kd(x, y)$. Show that f has a fixed point.

$^{+}$(b) Suppose instead we assume only that for all $x, y \in X$ we have $d(f(x), f(y)) \leq Kd(x, y)$, or $d(f(f(x)), f(f(y))) \leq Kd(x, y)$ or $d(f(f(f(x))), f(f(f(y)))) \leq Kd(x, y)$. Must f have a fixed point?