Mich. 2008 **ANALYSIS II—EXAMPLES 2** PAR

Unless stated otherwise, the space $\mathbb{R}^n$ may be assumed to have the Euclidean norm $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$, and the spaces ℓ_∞ and ℓ_0 the uniform norm $\|x\| = \sup_i |x_i|$.

1. Let $(x^{(m)})$ and $(y^{(m)})$ be sequences in $\mathbb{R}^n$ converging to x and y respectively. Show that $x^{(m)} \cdot y^{(m)} \to x \cdot y$. Deduce that if $f\colon \mathbb{R}^n \to \mathbb{R}^p$ and $g\colon \mathbb{R}^n \to \mathbb{R}^p$ are continuous at $x \in \mathbb{R}^n$ then so is the pointwise scalar product function $f \cdot g\colon \mathbb{R}^n \to \mathbb{R}$.

2. Show that $\|x\|_1 = \sum_{i=1}^n |x_i|$ defines a norm on $\mathbb{R}^n$. Show directly that it is Lipschitz equivalent to the Euclidean norm.

3. (a) Show that $\|f\|_1 = \int_0^1 |f|$ defines a norm on the vector space $C\big([0,1]\big)$. Is it Lipschitz equivalent to the uniform norm?
(b) Let $R\big([0,1]\big)$ denote the vector space of all integrable functions on $[0,1]$. Does $\|f\|_1 = \int_0^1 |f|$ define a norm on $R\big([0,1]\big)$?

4. Which of the following subsets of $\mathbb{R}^2$ are open? Which are closed? (And why?)
(i) $\{(x,0) : 0 \le x \le 1\}$;
(ii) $\{(x,0) : 0 < x < 1\}$;
(iii) $\{(x,y) : y \ne 0\}$;
(iv) $\{(x,y) : y = nx \text{ for some } n \in \mathbb{N}\} \cup \{(x,y) : x = 0\}$;
(v) $\{(x,y) : y = qx \text{ for some } q \in \mathbb{Q}\} \cup \{(x,y) : x = 0\}$;
(vi) $\{(x, f(x)) : x \in \mathbb{R}\}$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function;

5. Is the set $\{(x, x\sin\frac{1}{x}) : x \in (0,\infty)\} \cup \{(0,0)\}$ a path-connected subset of $\mathbb{R}^2$? What about the set $\{(x, \sin\frac{1}{x}) : x \in (0,\infty)\} \cup \{(0,0)\}$?

6. Is the set $\{f : f(1/2) = 0\}$ closed in the space $C\big([0,1]\big)$ with the uniform norm? What about the set $\{f : \int_0^1 f = 0\}$? In each case, does the answer change if we replace the uniform norm with the norm $\|\cdot\|_1$ defined in Q3?

7. Which of the following functions f are continuous?
(i) The linear map $f\colon \ell_\infty \to \mathbb{R}$ defined by $f(x) = \sum_{n=1}^\infty x_n/n^2$;
(ii) The identity map from the space $C\big([0,1]\big)$ with the uniform norm $\|\cdot\|$ to the space $C\big([0,1]\big)$ with the norm $\|\cdot\|_1$ as defined in Q3;
(iii) The identity map from $C\big([0,1]\big)$ with the norm $\|\cdot\|_1$ to $C\big([0,1]\big)$ with the uniform norm $\|\cdot\|$;
(iv) The linear map $f\colon \ell_0 \to \mathbb{R}$ defined by $f(x) = \sum_{i=1}^\infty x_i$.

8. For X, $Y \subset \mathbb{R}^n$, define $X + Y = \{x + y : x \in X, y \in Y\}$. Give examples of closed sets X, $Y \subset \mathbb{R}^n$ for some n such that $X + Y$ is not closed. Show that it is not possible to find such an example with X bounded. If V, $W \subset \mathbb{R}^n$ are open, must $V + W$ be open?

9. (a) Show that the space ℓ_∞ is complete. Show also that $c_0 = \{x \in \ell_\infty : x_n \to 0\}$, the vector subspace of ℓ_∞ consisting of all sequences converging to 0, is complete.
(b) Define a norm $\|\cdot\|$ on the space $R\big([0,1]\big)$ of Q3 by $\|f\| = \sup\{|f(x)| : x \in [0,1]\}$. Is it complete?

10. Let $\alpha\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that $\|x\|' = \|x\| + \|\alpha x\|$ defines a norm on $\mathbb{R}^n$. Using the fact that all norms on a finite-dimensional space are Lipschitz equivalent, deduce that α is continuous.

11. Let V be a normed space in which every bounded sequence has a convergent subsequence. Show that V must be complete. $^+$Show further that V must be finite-dimensional.

12. Let $(x^{(n)})_{n\geq 1}$ be a bounded sequence in ℓ_∞. Show that there is a subsequence $(x^{(n_j)})_{j\geq 1}$ which converges in every co-ordinate; that is to say, the sequence $(x_i^{(n_j)})_{j\geq 1}$ of real numbers converges for each i. Why does this not show that every bounded sequence in ℓ_∞ has a convergent subsequence?

13. Show that $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ defines a norm on ℓ_0, and that this norm is not Lipschitz equivalent to the uniform norm $\|\cdot\|$. Find a third norm on ℓ_0 which is equivalent neither to $\|\cdot\|_1$ nor to $\|\cdot\|$. Is it possible to find uncountably many norms on ℓ_0 such that no two are Lipschitz equivalent?

$^+$14. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function under which the image of any path-connected set is path-connected and the image of any closed bounded set is closed and bounded. Show that f must be continuous.