Mich. 2007 **ANALYSIS II—EXAMPLES 4** PAR

1. At which points are the following functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ differentiable?

(i) $f(x,y) = \begin{cases} x/y & y \neq 0 \\ 0 & y = 0 \end{cases}$;

(ii) $f(x,y) = |x||y|$;

(iii) $f(x,y) = xy|x-y|$;

(iv) $f(x,y) = \begin{cases} xy/\sqrt{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$;

(v) $f(x,y) = \begin{cases} xy\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

2. Let $\|\cdot\|$ denote the usual Euclidean norm on $\mathbb{R}^n$. Show that the map sending x to $\|x\|^2$ is differentiable everywhere. What is its derivative? Where is the map sending x to $\|x\|$ differentiable and what is its derivative?

3. We work in $\mathbb{R}^3$ with the Euclidean norm. Consider the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x) = x/\|x\|$ for $x \neq 0$, and $f(0) = 0$. Show that f is differentiable except at 0, and that

$$Df|_x(h) = \frac{h}{\|x\|} - \frac{x(x\cdot h)}{\|x\|^3}.$$

Verify that $Df|_x(h)$ is orthogonal to x and explain geometrically why this is the case.

4. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and $x \in \mathbb{R}^n$. For $e \in \mathbb{R}^n$ with $e \neq 0$, the *directional derivative of f at x in direction e* is defined to be

$$D_e f(x) = \lim_{h\to 0} \frac{f(x+he)-f(x)}{h},$$

when this limit exists. Show that if f is differentiable at x then the directional derivative $D_e f(x)$ exists for every $e \neq 0$. If the directional derivative $D_e f(x)$ exists for every $e \neq 0$, must f be differentiable at x?

5. Let $\mathcal{M}_n$ denote the space of $n \times n$ real matrices with the operator norm $\|\cdot\|$. Show that $\|AB\| \leq \|A\|\|B\|$ for all A, $B \in \mathcal{M}_n$.

6. Define $f\colon \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^2$. Show that f is differentiable everywhere and find its derivative.

7. Let $\mathcal{N}_n \subset \mathcal{M}_n$ be the set of invertible $n \times n$ matrices. Show that $\mathcal{N}_n$ is an open subset of $\mathcal{M}_n$.

Define $f\colon \mathcal{N}_n \to \mathcal{N}_n$ by $f(A) = A^{-1}$. Show that f is differentiable at the identity matrix I, and that $Df|_I(H) = -H$.

Let $B \in \mathcal{N}_n$ and define g_L, $g_R\colon \mathcal{N}_n \to \mathcal{N}_n$ by $g_L(A) = B^{-1}A$ and $g_R(A) = AB^{-1}$. Show that $f = g_R \circ f \circ g_L$ and hence, or otherwise, show that f is differentiable at B. What is $Df|_B$?

8. Define $\exp\colon \mathcal{M}_n \to \mathcal{M}_n$ by $\exp(A) = \sum_{n=0}^{\infty} A^n/n!$. Why is this function well-defined?

Show that exp is differentiable at 0. What is $D\exp|_0$?

Show that there is an open set $U \subset \mathcal{M}_n$ with $I \in U$ on which there is a well-defined logarithm; that is, there is a function $\log\colon U \to \mathcal{M}_n$ such that $\exp(\log(A)) = A$ for all $A \in U$. Show that log is differentiable at I. What is $D\log|_I$?

9. Show that $\det : \mathcal{M}_n \to \mathbb{R}$ is differentiable at the identity matrix I with $D\det|_I(H) = \operatorname{tr}(H)$. Deduce that det is differentiable at any invertible matrix A with $D\det|_A(H) = \det A \operatorname{tr}(A^{-1}H)$. Show further that det is twice differentiable at I and find $D^2\det|_I$ as a bilinear map from $\mathcal{M}_n \times \mathcal{M}_n$ to $\mathbb{R}$.

10. (a) Define $f\colon \mathcal{M}_n \to \mathcal{M}_n$ by $f(A) = A^3$. Find the Taylor series of $f(A+H)$ about A.

(b) Define $g\colon \mathcal{N}_n \to \mathcal{N}_n$ by $g(A) = A^{-1}$. Find the Taylor series of $g(I+H)$ about I.

11. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $(a,b) \in \mathbb{R}^2$.

(a) Suppose that $D_1 f$ exists and is continuous in some open ball around (a,b), and that $D_2 f$ exists at (a,b). Show that f is differentiable at (a,b).

(b) Suppose instead that $D_1 f$ exists and is bounded on some open ball around (a,b), and that for fixed x the function $y \mapsto f(x,y)$ is continuous. Show that f is continuous at (a,b).

12. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = xy(x^2-y^2)/(x^2+y^2)$ for $(x,y) \neq (0,0)$, and $f(0,0) = 0$. Show that

(i) f is continuous on $\mathbb{R}^2$;

(ii) the partial derivatives $D_1 f$ and $D_2 f$ exist and are continuous on $\mathbb{R}^2$; and

(iii) the partial derivatives $D_1 D_2 f$ and $D_2 D_1 f$ exist on $\mathbb{R}^2$.

Where are $D_1 D_2 f$ and $D_2 D_1 f$ continuous? Is $D_1 D_2 f(0,0) = D_2 D_1 f(0,0)$?

13. Let $U \subset \mathbb{R}^n$ be open and path-connected. Show that any two points $x, y \in U$ can be joined by a *polygonal path* in U, that is a path consisting of finitely many line-segments. Deduce the result that if $f\colon U \to R^m$ is differentiable on U with $Df|_x = 0$ for all $x \in U$ then f is constant.