Unless stated otherwise, the space $\mathbb{R}^n$ may be assumed to have the Euclidean norm $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$, and the spaces ℓ_∞ and ℓ_0 the uniform norm $\|x\| = \sup_i |x_i|$.

1. Let $(x^{(m)})$ and $(y^{(m)})$ be sequences in $\mathbb{R}^n$ converging to x and y respectively. Show that $x^{(m)}.y^{(m)} \to x.y$. Deduce that if $f\colon \mathbb{R}^n \to \mathbb{R}^p$ and $g\colon \mathbb{R}^n \to \mathbb{R}^p$ are continuous at $x \in \mathbb{R}^n$ then so is the function $f.g$ (where $f.g\colon \mathbb{R}^n \to \mathbb{R}$ is defined by $(f.g)(y) = f(y).g(y)$).

2. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function. Show that f is continuous at a point $x \in \mathbb{R}^n$ if and only if each of the co-ordinate functions $f_i\colon \mathbb{R}^n \to \mathbb{R}$ ($i = 1, 2, \dots, m$) is continuous at x. Give an example of a function $f\colon \mathbb{R} \to \ell_\infty$ such that each of the co-ordinate functions $f_i\colon \mathbb{R} \to \mathbb{R}$ ($i = 1, 2, 3, \dots$) is continuous but such that f itself is not continuous at some point.

[For a function f taking values in $\mathbb{R}^m$ (or in ℓ_∞), the co-ordinate functions f_i are the real-valued functions such that $f(x) = (f_1(x), f_2(x), \dots, f_m(x))$ (or $f(x) = (f_1(x), f_2(x), f_3(x), \dots)$) for each x.]

3. Show that $\|x\|_1 = \sum_{i=1}^n |x_i|$ defines a norm on $\mathbb{R}^n$. Show directly that it is Lipschitz equivalent to the Euclidean norm.

4. (a) Show that $\|f\|_1 = \int_0^1 |f|$ defines a norm on the vector space $C\big([0,1]\big)$. Is it Lipschitz equivalent to the uniform norm?

(b) Let $R\big([0,1]\big)$ denote the vector space of all integrable functions on $[0,1]$. Does $\|f\|_1 = \int_0^1 |f|$ define a norm on $R\big([0,1]\big)$?

5. Which of the following subsets of $\mathbb{R}^2$ are open? Which are closed? Which are path-connected? (And why?)
(i) $\{(x,0) : 0 \le x \le 1\}$;
(ii) $\{(x,0) : 0 < x < 1\}$;
(iii) $\{(x,y) : y \ne 0\}$;
(iv) $\{(x,y) : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$;
(v) $\{(x,y) : y = nx \text{ for some } n \in \mathbb{N}\} \cup \{(x,y) : x = 0\}$;
(vi) $\{(x,y) : y = qx \text{ for some } q \in \mathbb{Q}\} \cup \{(x,y) : x = 0\}$;
(vii) $\{(x, f(x)) : x \in \mathbb{R}\}$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function;
(viii) $\{(x, \sin\frac{1}{x}) : x \in (0,\infty)\} \cup \{(0,y) : -1 \le y \le 1\}$.

6. Is the set $\{f : f(1/2) = 0\}$ closed in the space $C\big([0,1]\big)$ with the uniform norm? What about the set $\{f : \int_0^1 f = 0\}$? In each case, does the answer change if we replace the uniform norm with the norm $\|\cdot\|_1$ defined in Q4?

7. Which of the following functions f are continuous?
(i) The linear map $f\colon \ell_\infty \to \mathbb{R}$ defined by $f(x) = \sum_{n=1}^\infty x_n/n^2$;
(ii) The identity map from the space $C\big([0,1]\big)$ with the uniform norm $\|\cdot\|_\infty$ to the space $C\big([0,1]\big)$ with the norm $\|\cdot\|_1$ as defined in Q4;
(iii) The identity map from $C\big([0,1]\big)$ with the norm $\|\cdot\|_1$ to $C\big([0,1]\big)$ with the norm $\|\cdot\|_\infty$;
(iv) The linear map $f\colon \ell_0 \to \mathbb{R}$ defined by $f(x) = \sum_{i=1}^\infty x_i$.

8. (a) Show that any linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ must be continuous.

(b) Show that $\|\alpha\| = \sup\{\|\alpha(x)\| : x \in \mathbb{R}^n, \|x\| \le 1\}$ defines a norm on the space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

9. For X, $Y \subset \mathbb{R}^n$, define $X + Y = \{x + y : x \in X, y \in Y\}$. Give examples of closed sets X, $Y \subset \mathbb{R}^n$ for some n such that $X + Y$ is not closed. Show that it is not possible to find such an example with X bounded. If V, $W \subset \mathbb{R}^n$ are open, must $V + W$ be open?

10. (a) Show that the space ℓ_∞ is complete. Show also that $c_0 = \{x \in \ell_\infty : x_n \to 0\}$, the vector subspace of ℓ_∞ consisting of all sequences converging to 0, is complete.

(b) Define a norm $\|\cdot\|_\infty$ on the space $R\big([0,1]\big)$ of Q4 by $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$. Is it complete?

11. Let V be a normed space in which every bounded sequence has a convergent subsequence.

(a) Show that V must be complete.

(b) Show further that V must be finite-dimensional

12. Show that a set $I \subset \mathbb{R}$ is path-connected if and only if I is an interval.

[An interval *is a set taking one of the forms* $(-\infty,\infty)$, $(-\infty,b)$, $(-\infty,b]$, $[a,\infty)$, (a,∞), $[a,b]$, $[a,b)$, $(a,b]$ *or* (a,b) *for some* a, $b \in \mathbb{R}$ *with* $a \le b$. *In particular, the empty set is an interval and each one-point set is an interval.]*

13. Let $(x^{(n)})_{n\ge1}$ be a bounded sequence in ℓ_∞. Show that there is a subsequence $(x^{(n_j)})_{j\ge1}$ which converges in every co-ordinate; that is to say, the sequence $(x_i^{(n_j)})_{j\ge1}$ of real numbers converges for each i. Why does this not show that every bounded sequence in ℓ_∞ has a convergent subsequence?

14. Show that $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ defines a norm on ℓ_0, and that this norm is not Lipschitz equivalent to the sup norm $\|\cdot\|_\infty$. Find a third norm on ℓ_0 which is equivalent neither to $\|\cdot\|_1$ nor to $\|\cdot\|_\infty$. Is it possible to find uncountably many norms on ℓ_0 such that no two are Lipschitz equivalent?

$^+$15. Let V be a real vector space with a countably infinite basis; that is to say, there is some sequence e_1, e_2, e_3, $\ldots$ of elements of V such that each non-zero $x \in V$ has a unique expression in the form $x = \sum_{i=1}^{k} \lambda_i e_{n_i}$ with k a positive integer, $1 \le n_1 < n_2 < \cdots < n_k$, and λ_1, λ_2, $\ldots$, λ_k non-zero real numbers. Show that any norm on V must be incomplete.

$^+$16. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function under which the image of any path-connected set is path-connected and the image of any closed bounded set is closed and bounded. Show that f must be continuous.