1. Which of the following sequences of functions converge uniformly on X?
(a) $f_n(x) = x^n$ on $X = [0, 1]$;
(b) $f_n(x) = \sin(n^2 x)/\log n$ on $X = \mathbb{R}$;
(c) $f_n(x) = x^n$ on $X = (0, 1)$;
(d) $f_n(x) = x^n$ on $X = (0, \frac{1}{2})$;
(e) $f_n(x) = xe^{-nx}$ on $X = [0, \infty)$;
(f) $f_n(x) = e^{-x^2} \sin(x/n)$ on $X = \mathbb{R}$.

2. Suppose that $f\colon [0, 1] \to \mathbb{R}$ is continuous. Show that the sequence $(x^n f(x))$ is uniformly convergent on $[0, 1]$ if and only if $f(1) = 0$.

3. Construct a sequence (f_n) of *continuous* real-valued functions on $[-1, 1]$ converging pointwise to the zero function but with $\int_{-1}^{1} f_n \not\to 0$. $^+$Is it possible to find such a sequence with $|f_n(x)| \leq 1$ for all n and for all x?

4. (a) Let (f_n) be a sequence of real-valued functions on a subset X of $\mathbb{R}$ converging uniformly to a function f. Suppose that each of the f_n is bounded. Show that f must be bounded.

(b) Give an example of a sequence (g_n) of bounded, real-valued functions on $[-1, 1]$ converging pointwise to a function g which is not bounded.

5. Let (f_n) and (g_n) be sequences of real-valued functions on a subset X of $\mathbb{R}$ converging uniformly to f and g respectively.

(a) Show that $f_n + g_n$ converges uniformly to $f + g$, and that λf_n converges uniformly to λf for each $\lambda \in \mathbb{R}$.

(b) Show that $f_n g_n$ need not converge uniformly to fg, but that if both f and g are bounded then $f_n g_n$ does converge uniformly to fg. What if f is bounded but g is not?

[Here and elsewhere, for functions f and g, the functions $f + g$ and fg are the pointwise sum and product respectively, i.e. $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$. For $\lambda \in \mathbb{R}$, the function λf is defined by $(\lambda f)(x) = \lambda f(x)$.]

6. Let (f_n) be a sequence of real-valued continuous functions on a closed, bounded interval $[a, b]$, and suppose that f_n converges pointwise to a continuous function f. Show that if $f_n \to f$ uniformly and (x_m) is a sequence of points in $[a, b]$ with $x_m \to x$ then $f_n(x_n) \to f(x)$. On the other hand, show that if f_n does not converge uniformly to f then we can find a convergent sequence $x_m \to x$ in $[a, b]$ such that $f_n(x_n) \not\to f(x)$.

7. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers.

(a) Define a sequence (f_n) of functions on $[-\pi, \pi]$ by $f_n(x) = \sum_{m=1}^{n} a_m \sin mx$. Show that each f_n is differentiable with $f_n'(x) = \sum_{m=1}^{n} m a_m \cos mx$.

(b) Show that $f(x) = \sum_{m=1}^{\infty} a_m \sin mx$ defines a continuous function on $[-\pi, \pi]$, but that the series $\sum_{m=1}^{\infty} m a_m \cos mx$ need not converge.

$^+$(c) Must f be differentiable? Give a proof or counterexample as appropriate.

8. Let f and g be uniformly continuous, real-valued functions on a subset X of $\mathbb{R}$.

(a) Show that $f+g$ is uniformly continuous, and that λf is uniformly continuous for each $\lambda \in \mathbb{R}$.

(b) Show that fg need not be uniformly continuous, but that if both f and g are bounded then fg is uniformly continuous. What if f is bounded but g is not?

9. Which of the following functions $f\colon [0,\infty) \to \mathbb{R}$ are (a) uniformly continuous; (b) bounded?
(i) $f(x) = \sin x^2$;
(ii) $f(x) = \inf\{|x-n^2| : n \in \mathbb{N}\}$;
(iii) $f(x) = (\sin x^3)/(x+1)$.

10. (a) Show that if (f_n) is a sequence of uniformly continuous, real-valued functions on $\mathbb{R}$, and if $f_n \to f$ uniformly, then f is uniformly continuous.

(b) Give an example of a sequence of uniformly continuous, real-valued functions (f_n) on $\mathbb{R}$ such that f_n converges pointwise to a function f which is continuous but not uniformly continuous.

11. Suppose that $f\colon [0,\infty) \to \mathbb{R}$ is continuous, and that $f(x)$ tends to a (finite) limit as $x \to \infty$. Must f be uniformly continuous on $[0,\infty)$? Give a proof or counterexample as appropriate.

12. Let f be a differentiable, real-valued function on $\mathbb{R}$, and suppose that f' is bounded. Show that f is uniformly continuous. *(You may wish to use the Mean Value Theorem.)*

Let $g\colon [-1,1] \to \mathbb{R}$ be the function defined by $g(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $g(0) = 0$. Show that g is differentiable, but that its derivative is unbounded. Is g uniformly continuous?

13. Does there exist an integrable function $f\colon [0,1] \to \mathbb{R}$ such that $f(x) > 0$ for all $x \in [0,1]$, but with $\int_0^1 f = 0$?

14. Let (f_n) be a sequence of continuous, real-valued functions on $[0,1]$ converging pointwise to a function f. Prove that there is some subinterval $[a,b]$ of $[0,1]$ with $a < b$ on which f is bounded.