Analysis II Example Sheet 4

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In several of the questions we work with the space $\mathcal{M}$ of all $n \times n$ matrices. Pick any appropriate norm: for example the operator norm or the Euclidean norm on the space viewed as $\mathbb{R}^{n^2}$.

1. Where are the following functions f from $\mathbb{R}^2$ to $\mathbb{R}$ differentiable?

$$f(x,y) = xy, \qquad f(x,y) = x/y, \qquad f(x,y) = |x||y|, \qquad f(x,y) = xy\,|x-y|$$

$$f(x,y) = xy/\sqrt{x^2+y^2} \quad ((x,y) \neq (0,0), \quad f(0,0) = 0$$
$$f(x,y) = xy \sin 1/x \quad (x \neq 0), \quad f(0,y) = 0$$

2. Let $\|\cdot\|$ denote the usual Euclidean norm on $\mathbb{R}^n$. Show that the map sending x to $\|x\|^2$ is differentiable everywhere. What is its derivative? Where is the map sending x to $\|x\|$ differentiable and what is its derivative?

3. We work in $\mathbb{R}^3$ with the Euclidean norm. Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by
$$f(x) = \frac{x}{\|x\|} \quad \text{for } x \neq 0$$
and with $f(0) = 0$. Show that f is differentiable except at 0 and
$$Df|_x(h) = \frac{h}{\|x\|} - \frac{x(x \cdot h)}{\|x\|^3}.$$
Verify that $Df(x)(h)$ is orthogonal to x and explain geometrically why this is the case.

4. Let $g : \mathbb{R} \to \mathbb{R}$ be a function satisfying $g(x+\pi) = -g(x)$. Show that all the directional derivatives of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(r,\theta) = rg(\theta)$ exist at zero. By choosing suitable g or otherwise, give an example of a function that:

   a) has all directional derivatives at zero but is not continuous at zero.

   b) has all directional derivatives at zero, is continuous everywhere, but is not differentiable at zero.

   c) has all directional derivatives at zero, is differentiable everywhere except zero, but is not differentiable at zero.

5. (i) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $D_1 f = \partial f/\partial x$ is continuous in a neighbourhood of (a,b), and $D_2 f = \partial f/\partial y$ exists at (a,b). Show that f is differentiable at (a,b).

   (ii) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $D_1 f = \partial f/\partial x$ exists and is bounded near (a,b), and that for a fixed, $f(a,y)$ is continuous as a function of y. Show that f is continuous at (a,b).

6. Let $\alpha\colon \mathcal{M} \to \mathcal{M}$ (see top of sheet for definition of $\mathcal{M}$) defined by $A \mapsto A^2$. Show that α is differentiable everywhere and find its derivative.

7. Let $f\colon \mathcal{M} \to \mathcal{M}$ be defined by $A \to A^{-1}$.

   Show that α is differentiable at the identity matrix and that $Df|_I(h) = -h$. Now suppose that B is invertible. Define the map $g_L : A \to B^{-1}A$ and $g_R : A \to AB^{-1}$. Show that $g_R f g_L(A) = A^{-1}$. Hence, find the derivative $Df|_B$.

8. Suppose that $B\colon \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}) \cong \mathbb{R}^m$ is differentiable at x_0 and that y is a fixed vector in $\mathbb{R}^m$. Let $\hat{y}$ be the function $L(\mathbb{R}^m, \mathbb{R}) \to \mathbb{R}$ sending a map α to the real number $\alpha(y)$.

   By using the chain rule or otherwise show that the map $g\colon \mathbb{R}^m \to \mathbb{R}$ defined by $g(x) = B(x)(y) = \hat{y}(B(x))$ (i.e., the linear map $B(x)$ evaluated at y) is differentiable and find its derivative.

   Now suppose $f\colon \mathbb{R}^m \to \mathbb{R}$ is twice differentiable. Deduce that the map $g\colon \mathbb{R}^m \to \mathbb{R}$ sending x to $Df|_x(h)$ is differentiable with derivative $D\alpha|_x(k) = D^2 f|_x(k)(h)$.

9. Consider the map $f : \mathcal{M} \to \mathcal{M}$ defined by $f(A) = A^3$. Find the Taylor Series of $f(A+h)$.

10. Suppose $f(0,0) = 0$, and

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2} \qquad \text{if } (x,y) \neq (0,0).$$

   Prove that

   (a) f, $D_1 f$, $D_2 f$ are continuous in $\mathbb{R}^2$;

   (b) $D_1 D_2 f$ and $D_2 D_1 f$ exist at every point in $\mathbb{R}^2$;

   (c) $D_1 D_2 f(0,0) = 1$ and $D_2 D_1 f(0,0) = -1$.

   Where are $D_1 D_2 f$ and $D_2 D_1 f$ continuous?

11. Suppose that X is a complete metric space, and that α, β are contractions on X with Lipschitz constant at most $1/2$, and with fixed points a and b respectively. Suppose that $\sup_{x \in X} d(\alpha(x), \beta(x)) < \varepsilon$. Show that $d(a,b) < 2\varepsilon$.

   Deduce that the inverse function defined in the proof of the inverse function theorem is continuous.

12. Consider the function $f : \mathcal{M} \to \mathcal{M}$ defined by $A \to \exp(A) = \sum_{n=0}^{\infty} A^n/n!$. Find the derivative $Df|_0$. Show that there is an open set $U \subset \mathcal{M}$ containing I such that "log" is well defined: i.e., there is a function $l : U \to \mathcal{M}$ such that $\exp(l(A)) = A$ for all $A \in U$. What is the derivative $Dl|_I$?

13. Give an example of a continuous bijection from $\mathbb{R}$ to $\mathbb{R}$ which has a point x with $Df|_x = 0$. (Harder) can $Df|_x = 0$ hold for uncountably many points x?

14. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(0) = 0$ and f is differentiable at 0 with $f'(0) \neq 0$. Show that there need not be any neighbourhood of 0 on which f is injective. Can such an f be differentiable everywhere?

15. Suppose that $f\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, f is continuously differentiable, $x_0 \in \mathbb{R}^m$, $y_0 \in \mathbb{R}^n$, that $f(x_0, y_0) = 0$ and the map $\bar{f}(\cdot) = f(x, \cdot)$ has non-singular derivative at y_0. Let S be the set $\{(x,y) : f(x,y) = 0\}$ considered as a metric space in its own right.

   By considering the function $\hat{f}\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ defined by $\hat{f}(x,y) = (x, f(x,y))$ or otherwise show that there exists an open set $U \subset \mathbb{R}^m$ with $x_0 \in U$ and V an open subset of the metric space S and a continuous bijection with continuous inverse $g : U \to V$.

16. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^2 + y^2 - 1$. Let $V = \{(x,y) : f(x,y) = 0\}$. Using the previous question show that every point of V lies in an open set U which is homeomorphic to a subset of $\mathbb{R}$.