Analysis II Example Sheet 3

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1. For each of the following sets X, determine whether the given function d defines a metric on X:

   (i) $X = \mathbb{R}^n$, $d(x, y) = \min\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.

   (ii) $X = \mathbb{Z}$, $d(x, x) = 0$ for all x, otherwise $d(x, y) = 2^n$ if $x - y = 2^n a$ where a is odd.

   (iii) $X = \mathbb{Q}$, $d(x, x) = 0$ for all x, otherwise $d(x, y) = 3^{-n}$ if $x - y = 3^n a/b$ where a, b are prime to 3 (and n may be positive, negative or zero).

   (iv) $X = \{\text{functions } \mathbb{N} \to \mathbb{N}\}$, $d(f, f) = 0$, otherwise $d(f, g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

   (v) $X = \mathbb{C}$, $d(z, w) = |z - w|$ if z and w are on the same straight line through 0, otherwise $d(z, w) = |z| + |w|$.

2. Let d be the normal metric on $\mathbb{R}$ and let d' be the discrete metric on $\mathbb{R}$ (that is $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$). Show that all functions $f\colon (\mathbb{R}, d') \to (\mathbb{R}, d)$ are continuous. What are the continuous functions from $(\mathbb{R}, d) \to (\mathbb{R}, d')$.

3. In the metric space defined in Q1 part (iii) does the sequence $x_n = 3^n$ converge? What about $y_n = \sum_{i=0}^{n} 3^i$? And $z_n = \sum_{i=0}^{n} 3^{i^2}$? Are they Cauchy? Is this metric space complete?

4. Let (X, d) be a metric space. Show that

$$d_1(x, y) = \min(1, d(x, y)) \quad \text{and} \quad d_2(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

   are metrics on X topologically equivalent to d. Are the metrics d, d_1 and d_2 uniformly equivalent? Are they Lipschitz equivalent?

5. Suppose that (X, d_X) and (Y, d_Y) are metric spaces and that d_Y is bounded: i.e., $d_Y(y, y') < M$ for all $y, y' \in Y$. Show that the set of functions from $X \to Y$ with distance D defined by $D(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$ is a metric space.

6. Suppose that a metric d on a set X satisfies the following stronger form of the triangle inequality:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} \quad \text{for all } x, y, z \in X .$$

   Show that every open ball in X is also a closed set. Does it follow that every open set must be closed? Give an example of such a metric space.

7. (i) Show that the space of real sequences $\mathbf{a} = (a_n)_{n=1}^{\infty}$ with all but finitely many of the a_n are zero is not complete in the norm defined by $\|\mathbf{a}\|_1 = \sum_{n=1}^{\infty} |a_n|$. Is there an obvious way of 'completing' the space?

   (ii) Let $\|\cdot\|_1$ be the norm on the space of the continuous functions on $[0, 1]$ defined by $\|f\|_1 = \int_0^1 |f|$ (see sheet 2 Q2). Is this norm complete?

8. Let X be the space of bounded real sequences. Is there a metric on X such that a sequence of vectors $x^{(n)} \to x$ in the metric if and only if $x^{(n)}$ converges to x coordinatewise? Is there a norm with this property?

9. Let (X, d) be a metric space. Let $C(X)$ denote the space of bounded continuous functions from $X \to \mathbb{R}$ with norm $\|f\| = \sup_{x \in X} |f(x)|$. Show carefully that the space $C(X)$ is complete in this norm. [Hint: we saw in lectures that the space $C([0, 1])$ with the sup norm is complete.]

10. Show that $x = \cos x$ has a unique solution. Use a reasonable pocket calculator to find the solution to some decimal places. (This should take no time. Remember to work in radians!)

11. Find a linear map α from $\mathbb{R}^2$ to $\mathbb{R}^2$ that is a contraction in the usual Euclidean norm but not in the norm $\|(x, y)\|_\infty = \max |x|, |y|$.

12. Let $I = [0, R]$ be an interval and let $C(I)$ be the space of continuous functions on I. Show that, for any α the norm $\|f\| = \sup_{x \in I} \|f(x)e^{-\alpha x}\|$ is an equivalent norm to the usual sup norm.

    Now suppose that $\phi : \mathbb{R}^2 \to \mathbb{R}$ is continuous and Lipschitz in the second variable. Show that there exists a norm on $C(I)$ such that the map sending f to $y_0 + \int_0^x \phi(t, f(t))dt$ is a contraction. Deduce that the differential equation $f'(x) = \phi(x, f(x))$ has a unique solution on I satisfying $f(0) = y_0$.

13. Let X, d be a metric space and suppose that $f \colon X \to X$ and for some n the function f^n has a unique fixed point (where f^n denotes the function f applied n times). Prove that f has a unique fixed point.

14. Suppose that X is a closed and bounded subset of $\mathbb{R}^n$, and that $f \colon X \to X$ is a map such that $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$ where d is the metric inherited from the usual Euclidean norm on $\mathbb{R}^n$. Must f have a fixed point?

15. [Tripos IB 93301(b)] Let (X, d) be a metric space without isolated points (i.e. such that $\{x\}$ is not open for any $x \in X$), and $(x_n)_{n \geq 0}$ a sequence of points of X. Show that it is possible to find a sequence of points y_n of X and positive real numbers r_n such that $r_n \to 0$, $d(x_n, y_n) > r_n$ and

    $$B(y_n, r_n) \subseteq B(y_{n-1}, r_{n-1})$$

    for each $n > 0$. Deduce that a nonempty complete metric space without isolated points has uncountably many points.

16. Suppose that (X, d) is a metric space. Must there exist a subset of a normed space isometric to X? (I.e., must there exist a distance preserving map from X into a normed space?)

17. Let X be the space of continuous functions on $[0, 1]$. Is there a metric on X such that a sequence of functions $f_n \to f$ in the metric if and only if f_n converges to f pointwise?