Analysis II Example Sheet 2

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Unless otherwise specified the space $\mathbb{R}^n$ may be assumed to have the Euclidean norm $\|x\| = \|x\|_2 = \sqrt{\sum_i x_i^2}$ and the space $C([0,1])$ the sup norm $\|f\| = \|f\|_\infty = \sup_x |f(x)|$. The space l_∞ is the space of all bounded real sequences $(x_i)_{i=1}^\infty$ with norm $\|x\| = \sup_i |x_i|$.

1. Show that $\|x\|_1 = \sum_{i=1}^n |x_i|$ is a norm on $\mathbb{R}^n$. Show explicitly that it is Lipschitz Equivalent to $\|\cdot\|_2$.

2. a) Show carefully that the vector space of continuous functions on $[0,1]$ with norm $\|f\|_1 = \int_0^1 |f|$ is a normed space.

   b) Is this norm equivalent to the uniform norm $\|\cdot\|_\infty$?

   c) Is the space of all (Riemann) integrable functions with norm $\|f\| = \int_0^1 |f|$ a normed space?

3. Which of the following subsets of $\mathbb{R}^2$ are (a) open, (b) closed? (And why?)

   (i) $\{(x,0) : 0 \le x \le 1\}$ (ii) $\{(x,0) : 0 < x < 1\}$ (iii) $\{(x,y) : y \ne 0\}$
   (iv) $\{(x,y) : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$ (v) $\{(x,y) : xy = 1\}$ (vi) $\{(x,y) : y = nx,\ n \in \mathbb{N}\} \cup \{(x,y) : x = 0\}$

   (vii) $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ where (x_n) is a sequence in $\mathbb{R}^2$ converging to x.

   (viii) $\{(x, f(x)) : x \in \mathbb{R}\}$ where $f \colon \mathbb{R} \to \mathbb{R}$ is a continuous function.

4. Which of the following sets are a) open? b) closed?

   (i) $\{0\}$ in an arbitrary normed space X,
   (ii) The whole space X in an arbitrary normed space X,
   (iii) $\{x :$ there exists N such that $x_n = 0$ for all $n > N\}$ in l_∞,
   (iv) $\{x : x_n \to 0\}$ in l_∞,
   (v) $\{f : f(1/2) = 0\}$ in $C([0,1])$ with the sup norm $\|\cdot\|_\infty$,
   (vi) $\{f : \int_0^1 f = 0\}$ in $C([0,1])$ with the sup norm $\|\cdot\|_\infty$,
   (vii) $\{f : f(1/2) = 0\}$ in $C([0,1])$ with the norm $\|\cdot\|_1$ defined in Q2,
   (viii) $\{f : \int_0^1 f = 0\}$ in $C([0,1])$ with the norm $\|\cdot\|_1$.

5. Which of the following functions f are continuous?

   (i) The linear map $f \colon l_\infty \to \mathbb{R}$ defined by $f(x) = \sum_{n=1}^\infty x_n/n^2$,
   (ii) The identity map from $C([0,1], \|\cdot\|_\infty)$ to $C([0,1], \|\cdot\|_1$ (where $\|f\|_1 = \int_0^1 |f|$ as in question 2)
   (iii) The identity map from $C([0,1], \|\cdot\|_1)$ to $C([0,1], \|\cdot\|_\infty)$
   (iv) Let c_{00} denote the subspace of l_∞ consisting of all sequences with finitely many non zero terms. Define f to be the linear map $f(x) = \sum_{i=1}^\infty x_i$.

6. Let $\alpha\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, say given by $x \to Ax$ where A is an $m \times n$ matrix. Show that α is continuous.

7. Let $\alpha\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Let

$$\|\alpha\| = \sup\{\|\alpha(\mathbf{x})\| : \mathbf{x} \in \mathbb{R}^n,\ \|\mathbf{x}\| \leq 1\}.$$

Show that the function which sends α to $\|\alpha\|$ is a norm on the vector space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ of all linear maps $\mathbb{R}^n \to \mathbb{R}^m$. [This is the *operator norm* on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.] Show also that

$$\|\alpha\| = \sup\{\|\alpha(\mathbf{x})\| : \mathbf{x} \in \mathbb{R}^n,\ \|\mathbf{x}\| = 1\} = \sup\{\|\alpha(\mathbf{x})\|/\|\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n,\ \mathbf{x} \neq \mathbf{0}\}\,.$$

8. Suppose that V and W are normed spaces and that $\alpha\colon V \to W$ is linear. Prove that $\|x\|' := \|x\| + \|\alpha(x)\|$ is a norm on V. Hence, or otherwise, show that if V is finite dimensional then α is continuous.

9. In lectures we proved that if K is a closed and bounded set in $\mathbb{R}^d$, then any continuous function defined on K has bounded image. Prove the converse: if every continuous real-valued function on $K \subseteq \mathbb{R}^d$ is bounded, then K is closed and bounded.

10. Suppose that A, B are subsets of $\mathbb{R}^n$. Define the distance between A and B to be $d(A, B) = \inf_{a\in A, b\in B} d(a, b)$. Show that there exist disjoint closed subsets A, B of $\mathbb{R}$ (or $\mathbb{R}^2$ if you prefer) with $d(A, B) = 0$. Show that this cannot happen if A is bounded.

11. Suppose that A in a subset of a normed space and that A has the Bolzano-Weierstrass property. Show that A is complete.

12. Show that the space l_∞ is complete. Let $c_0 = \{x : x_n \to 0\}$ in l_∞. Show that c_0 is complete.

13. Suppose that $(x^{(n)})$ is a bounded sequence of vectors in l_∞. Show that there is a subsequence $x^{(n_i)}$ such that every coordinate is convergent: i.e., for every j the sequence $x_j^{(n_i)}$ of real numbers is convergent. Why does this not show that l_∞ has the Bolzano-Weierstrass property?

    Indeed, show that l_∞ does not have the Bolzano-Weierstrass property.

14. Find a countable subset X of $\mathbb{R}$ that is *dense*, that is meets every open subset of $\mathbb{R}$. Using uniform continuity or otherwise find a countable dense subset of $C([0, 1])$ with the sup norm $\|\cdot\|_\infty$. Does there exist a countable dense subset of l_∞, the space of bounded real sequences with the sup norm?

15. A subset K of $C([0, 1])$ is said to be equicontinuous if for all $x \in [0, 1]$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all y with $|x - y| < \delta$ *and for all* $f \in K$. Prove that every sequence (f_n) in K has a convergent subsequence (in the sup norm $\|\cdot\|_\infty$) if and only if K is equicontinuous and bounded. (Note the subsequence is not required to converge to a function in K.)