Analysis II Example Sheet 1

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1. Which of the following sequences of functions converge uniformly on X? (N.B. These are sequences not sums.)

   a) $f_n(x) = x^{1/n}$ where $X = [0, 1]$;

   b) $f_n(x) = \frac{1}{\log n} \sin(n^2 x)$ where $X = \mathbb{R}$;

   c) $f_n(x) = x^n$ where $X = (0, 1)$;

   d) $f_n(x) = x^n$ where $X = (0, 1/2)$;

   e) $f_n(x) = xe^{-nx}$ where $X = [0, \infty)$;

   f) $f_n(x) = e^{-x^2} \sin(x/n)$ where $X = \mathbb{R}$.

2. Suppose that $f\colon [0, 1] \to \mathbb{R}$ is continuous. Show that the sequence $x^n f(x)$ is uniformly convergent on $[0, 1]$ if and only if $f(1) = 0$.

3. Suppose that (f_n) is a sequence of real valued functions on a subset X of $\mathbb{R}$ converging uniformly to f.

   a) Suppose that each of the f_n is bounded. Show that f is bounded.

   b) Suppose that f is bounded. Prove that there exists M and n_0 such that $|f_n(x)| < M$ for all $n > n_0$ and for all x.

4. Suppose that (f_n) and (g_n) are sequences of real valued functions on a subset X of $\mathbb{R}$ converging uniformly to f and g respectively. Show that $f_n g_n$ need not converge uniformly to fg. However show that it if both f and g are bounded then $f_n g_n \to fg$ uniformly.

5. Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers and that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

   a) Prove that
$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx$$
defines a continuous function on $[-\pi, \pi]$.

   b) Guess an expression for f'. Must your guess converge?

   c$^+$) Must f be differentiable? Give a proof or counterexample.

6. Suppose f_n is a sequence of continuous functions from a bounded closed interval $[a, b]$ to $\mathbb{R}$, and that f_n converges pointwise to a continuous function f.

   a) If f_n converges uniformly to f, and (x_m) is a sequence of points of $[a, b]$ converging to a limit x, show that $f_n(x_n) \to f(x)$. [Careful – this is not quite as easy as it looks!]

   b) If f_n does not converge uniformly, show that we can find a convergent sequence $x_n \to x$ in $[a, b]$ such that $f_n(x_n)$ does not converge to $f(x)$.

7. Which of the following functions f are (a) uniformly continuous, (b) bounded on $[0,\infty)$?

   (i) $f(x) = \sin x^2$.

   (ii) $f(x) = \inf\{|x-n^2| : n \in \mathbb{N}\}$.

   (iii) $f(x) = (\sin x^3)/(x+1)$.

8. Suppose that f is continuous on $[0,\infty)$ and that $f(x)$ tends to a (finite) limit as $x \to \infty$. Is f necessarily uniformly continuous on $[0,\infty)$? Give a proof or a counterexample as appropriate.

9. a) Show that if (f_n) is a sequence of uniformly continuous functions on $\mathbb{R}$, and $f_n \to f$ uniformly on $\mathbb{R}$, then f is uniformly continuous.

   b) Give an example of a sequence of uniformly continuous functions f_n on $\mathbb{R}$, such that f_n converges pointwise to a continuous function f, but f is not uniformly continuous.

10. a) Suppose that f is a differentiable function on $\mathbb{R}$ such that f' is bounded. Use the Mean Value Theorem to show that f is uniformly continuous.

    b) Show that $f\colon [-1,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 \text{ if } x = 0 \\ x^2 \sin(1/x^2) \text{ if } x \neq 0 \end{cases}$$

    is differentiable on $[-1,1]$ but its derivative is unbounded. Is f uniformly continuous?

11. Suppose that f is continuous from $[0,1]\times[0,1]$ to $\mathbb{R}$. Prove that f is uniformly continuous. (M&T only) Prove that any continuous function from a compact metric space to $\mathbb{R}$ is uniformly continuous.

12. Integrate the following complex valued functions from $[0,1] \to \mathbb{C}$ in two ways; first do it directly i.e., by finding an antiderivative, secondly do it by splitting into real and imaginary parts and doing each separately. Check that your answers are the same in each case.

    a) $f(x) = x$

    b) $f(x) = \exp(x)$

    c) $f(x) = x + ix^2$

13. Give an example of a function that is continuous on $[-1,1]$ but not differentiable at zero.

    a) By writing $f = \sum_n f_n$ for a suitable sequence of functions (f_n), or otherwise, construct a function which is continuous on $[-1,1]$ but has infinitely many points at which it is not differentiable.

    b) Construct a continuous function f on $[-1,1]$ which is not differentiable at any point of $\mathbb{Q}$.

    c$^+$) Does there exist a continuous function on $[-1,1]$ which is not differentiable anywhere?

14. $^+$ Suppose that $f : [0,1]^2 \to \mathbb{R}$ is a function and that for each $y \in [0,1]$ the integral $\int_0^1 f(x,y)dx$ exists and for each $x \in [0,1]$ the integral $\int_0^1 f(x,y)dy$ exists. Give an example such that $\int_0^1\int_0^1 f(x,y)dxdy \neq \int_0^1\int_0^1 f(x,y)dydx$.

    Prove that this cannot happen if f is continuous.

15. $^+$ Suppose that f_n are continuous functions from $[0,1]$ to $[0,1]$ such that $f_n \to 0$ pointwise. Must $\int_0^1 f_n \to 0$?