ANALYSIS II EXAMPLES 4

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These Basic Questions cover the last quarter of the course. There are a few changes from last year. Since general topological phenomena are now treated in Metric and Topological Spaces, it seemed right to put more stress on applications. There are too many Additional Questions: but they should not do you any harm.

Comments and corrections are welcome and may be e-mailed to me at `m.hyland@dpmms.cam.ac.uk`.

Basic Questions

1. For each of the following sets X, determine whether the given function d defines a metric on X:

(i) $X = \mathbb{R}^n$, $d(x, y) = \min\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.

(ii) $X = \mathbb{Z}$, $d(x, x) = 0$ for all x, otherwise $d(x, y) = 2^n$ if $x - y = 2^n a$ where a is odd.

(iii) $X = \mathbb{Q}$, $d(x, x) = 0$ for all x, otherwise $d(x, y) = 3^{-n}$ if $x - y = 3^n a/b$ where a, b are prime to 3 (and n may be positive, negative or zero).

(iv) $X = \{$functions $\mathbb{N} \to \mathbb{N}\}$, $d(f, f) = 0$, otherwise $d(f, g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

(v) $X = \mathbb{C}$, $d(z, w) = |z - w|$ if z and w are on the same straight line through 0, otherwise $d(z, w) = |z| + |w|$.

2. Let (X, d) be a metric space. Show that

$$d_1(x, y) = \min(1, d(x, y)) \quad \text{and} \quad d_2(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

are metrics on X topologically equivalent to d. Are the metrics d, d_1 and d_2 uniformly equivalent? Are they Lipschitz equivalent?

3. (i) Suppose that $A \subseteq X$ is a subset in a metric space (X, d). Show that

$$d(x, A) = \inf_{a \in A} d(x, a)$$

defines a continuous real-valued function on X.

(ii) Suppose that A and B are disjoint closed sets in a metric space (X, d). Show that there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

(iii) Give an example of disjoint closed sets A and B in $\mathbb{R}^2$ with points $a_n \in A$ and $b_n \in B$ such that $d(a_n, b_n) \to 0$ as $n \to \infty$. (So disjoint closed sets need not be a finite distance apart.)

4. Suppose that (X, d_X) is a metric space. For $Y \subseteq X$ let (Y, d_Y) be the induced metric subspace.

(i) Show that U is open in (Y, d_Y) if and only if $U = Y \cap V$ for some V open in (X, d_X).

(ii Show that A is closed in (Y, d_Y) if and only if $A = Y \cap B$ for some B closed in (X, d_X).

5. (i) Show that the space of real sequences $\mathbf{a} = (a_n)_{n=1}^\infty$ with all but finitely many of the a_n are zero is not complete in the norm defined by $\|\mathbf{a}\|_1 = \sum_{n=1}^\infty |a_n|$. Is there an obvious 'completion'?

(ii) Show that $d(f, g) = \int_a^b |f(x) - g(x)| \, dx$ is a metric on $\mathcal{C}[a, b]$ the space of continuous functions on $[a, b]$. Is $(\mathcal{C}[a, b], d)$ complete?

6. Show that $x = \cos x$ has a unique solution. Use a reasonable pocket calculator to find the solution to some decimal places. (This should take no time. Remember to work in radians!)

7. Consider the map $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x^3 - 3x^2 + 3x$. For which initial values $x_0 \in \mathbb{R}$ does the sequence of iterates $x_n = T^n(x_0)$ converge and to what value? (Clearly it will be helpful to sketch a graph; but you should give proofs!)

8. Suppose that (X, d) is a (not necessarily complete) metric space and that $T : X \to X$ is a contraction.

(i) Show that if T has a fixed point, then it is unique.

(ii) Show that for any choice of x_0, if the sequence x_n defined by setting $x_{n+1} = Tx_n$ converges, then it converges to a fixed point of T.

(iii) Show that if T has a fixed point, then for any choice of x_0, the sequence x_n defined by setting $x_{n+1} = Tx_n$ converges to the fixed point.

(iv) Give an example of a non-empty metric space (X, d) and contraction T with no fixed point.

9. [Tripos IB 96401(b), modified] (i) Suppose that (X, d) is a nonempty complete metric space, and $f : X \to X$ a continuous map such that, for any $x, y \in X$, the sum $\sum_{n=1}^{\infty} d(f^n(x), f^n(y))$ converges. Show that f has a unique fixed point.

(ii) By considering the function $x \mapsto \max\{x - 1, 0\}$ on the interval $[0, \infty) \subseteq \mathbb{R}$, show that a function satisfying the hypotheses of (i) need not be a contraction mapping.

(iii) Let ϕ be a continuous real-valued function on $\mathbb{R} \times [a, b]$ which satisfies the Lipschitz condition

$$|\phi(x, t) - \phi(y, t)| \le M\,|x - y|\ ,\ \text{ for all } x, y \in \mathbb{R} \text{ and } t \in [a, b],$$

and let $g \in \mathcal{C}[a, b]$. Define $F : \mathcal{C}[a, b] \to \mathcal{C}[a, b]$ by $F(h)(t) = g(t) + \int_a^t \phi(h(s), s)\,ds$. Show by induction that

$$|F^n(h)(t) - F^n(k)(t)| \le \frac{1}{n!} M^n (t - a)^n \|h - k\|_\infty\ ,$$

for all $h, k \in \mathcal{C}[a, b]$ and $a \le t \le b$, and deduce that F has a unique fixed point.

(iv) In the original 1996 Tripos question from which this question was adapted, the word 'continuous' in the second line of part (i) was accidentally omitted. Give a counterexample to the result which the 1996 IB students were asked to prove.

10. [Tripos IB 95401(b)] For which a and b, with $a \le 0 \le b$, is the mapping $T : \mathcal{C}[a, b] \to \mathcal{C}[a, b]$ defined by

$$T(f)(x) = 1 + \int_0^x 2t\,f(t)\,dt$$

a contraction? Deduce that the differential equation

$$\frac{dy}{dx} = 2xy\ ,\quad \text{with } y = 1 \text{ when } x = 0\ ,$$

has a unique solution in some interval containing 0. In what interval can the differential equation be solved?

11. For fixed $B \in \mathbf{M}_n(\mathbb{R})$ define $f_B : \mathbf{M}_n \to \mathbf{M}_n$ by $f_B(X) = X - \exp(X) + B$.
Show directly that if $\|X\| \le K$ and $\|Y\| \le K$, then $\|f_B(X) - f_B(Y)\| \le \|X - Y\|(e^K - 1)$.
Show that if $\|I - B\| \le 1/5$ then f_B maps the ball $\{X : \|X\| \le 1/3\}$ into itself, and deduce that the equation $\exp(X) = B$ has a solution for $\|X\| \le 1/3$.

12. Consider the map $\Phi : \mathbf{M}_n(\mathbb{R}) \to \mathbf{M}_n(\mathbb{R})$ defined by $\Phi(A) = A.A^t$. Show that the image of Φ is included in a subspace of dimension $\frac{1}{2}n(n - 1)$. What is the derivative of Φ? What is the rank of $\Phi'(I)$, the derivative of Φ at the identity matrix I?

Let $\mathbf{O}_n(\mathbb{R})$ be the group of orthogonal transformations in n dimensions. Show that there is an open neighbourhood U of $I \in \mathbf{O}_n(\mathbb{R})$ and a differentiable map from a $\frac{1}{2}n(n - 1)$-dimensional Euclidean open ball V to $\mathbf{M}_n$ which is bijective onto U. Deduce the same result for an arbitrary $P \in \mathbf{O}_n(\mathbb{R})$.

Let $\mathbf{so}_n(\mathbb{R})$ be the space of skew symmetric real matrices. What is its dimension? Can you describe an explicit differentiable map from a open neighbourhood of $\mathbf{0}$ in $\mathbf{so}(n)$ to an open neighbourhood of I in $\mathbf{O}_n(\mathbb{R})$?

Additional Questions

13. A metric d on a set X is called an *ultrametric* if it satisfies the following stronger form of the triangle inequality:

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \quad \text{for all } x, y, z \in X \ .$$

Which of the metrics in question 1 are ultrametrics? Show that in an ultrametric space 'every triangle is isosceles' (that is, at least two of $d(x,y)$, $d(y,z)$ and $d(x,z)$ must be equal), and deduce that every open ball in an ultrametric space is a closed set. Does it follow that every open set must be closed?

14. There is a persistent 'urban myth' about the mathematics research student who spent three years writing a thesis about properties of 'antimetric spaces', where an *antimetric* on a set X is a function $d \colon X \times X \to \mathbb{R}$ satisfying the same axioms as a metric except that the triangle inequality is reversed (i.e. $d(x,z) \geq d(x,y) + d(y,z)$ for all x, y, z). Why would such a thesis be unlikely to be considered worth a Ph.D.?

15. Show that the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} x & \text{if } y \text{ is irrational,} \\ 2x & \text{if } y \text{ is rational,} \end{cases}$$

takes open sets to open sets, but is not continuous.

16. Suppose (X, d_X) and $Y, d_Y)$ are metric spaces and $f : X \to Y$ is a map between them.

(i) Let $\{U_i \ : \ i \in I\}$ be a family of open subsets of X with $X = \bigcup\{U_i \ : \ i \in I\}$. Show that f is continuous if and only if all the restrictions $f_i : U_i \to Y$ of f to U_i are continuous.

(ii) Let A and B be closed subsets of X with $A \cup B = X$. Show that f is continuous if and only if all the restrictions $f_A : A \to Y$ and $f_B : B \to Y$ of f to A and B respectively are continuous.

17. (i) Show that $\mathbf{GL}_n(\mathbb{R})$, the collection (group) of all invertible matrices, is an open subset of the space $\mathbf{M}_n(\mathbb{R})$ of all matrices.
[*Hint: Show that* $\mathbf{GL}_n(\mathbb{R})$ *is the inverse image under a continuous map of some (simple) open set.*]

(ii) Show that $\mathbf{O}_n(\mathbb{R})$, the collection (group) of all orthogonal matrices is a closed subset of the space $\mathbf{M}_n(\mathbb{R})$ of all matrices.
[*Hint: Show that* $\mathbf{O}_n(\mathbb{R})$ *is the inverse image under a continuous map of some (simple) closed set.*]

18. Suppose that $f : X \to Y$ is a continuous map. Consider the graph $G_f \subseteq X \times Y$ of f defined by

$$G_f = \{(x,y) \ : \ f(x) = y\} \ .$$

(i) Show that G_f is closed in $X \times Y$ (equipped with a product metric).

(ii) Show that the subspace G_f of $X \times Y$ is homeomorphic to X.

19. [Tripos IB 93301(b)] Let (X, d) be a metric space without isolated points (i.e. such that $\{x\}$ is not open for any $x \in X$), and $(x_n)_{n \geq 0}$ a sequence of points of X. Show that it is possible to find a sequence of points y_n of X and positive real numbers r_n such that $r_n \to 0$, $d(x_n, y_n) > r_n$ and

$$B(y_n, r_n) \subseteq B(y_{n-1}, r_{n-1})$$

for each $n > 0$. Deduce that a nonempty complete metric space without isolated points has uncountably many points. [This is a direct generalization of a proof of the uncountability of $\mathbb{R}$.]

20. Suppose that (X, d) is a complete metric space. Let $U \subset X$ be a proper open subset with complement $A = X - U$. Show the following.

(i) $d(x, A) = \inf\{d(x, a) \ : \ a \in A\}$ is a continuous real-valued function on X.

(ii) $f(x) = (d(x, A))^{-1}$ is a continuous real-valued function on U.

(iii) $\bar{d}(x, y) = d(x, y) + |f(x) - f(y)|$ is a metric on U.

(iv) d and $\bar{d}$ are equivalent metrics on U, but are not in general uniformly equivalent.

(v) $(U, \bar{d})$ is a complete metric space.

21. Is $T : C[0,1] \to C[0,1]$; $Tf(x) = \int_0^x f(t^2)dt$ a contraction mapping? Show that T has a unique fixed point.

22. Suppose (X,d) is a metric space and $T : X \to X$ satisfies $d(Tx,Ty) < d(x,y)$ for all $x \neq y$ in X.

   (i) Show that if T has a fixed point, then it is unique.

   (ii) Show that for any choice of x_0, if the sequence x_n defined by setting $x_{n+1} = Tx_n$ converges, then it converges to a fixed point of T.

   (iii) Give an example of a metric space (X,d) and map $T : X \to X$, satisfying $d(Tx,Ty) < d(x,y)$ for all $x \neq y$ in X, and with a fixed point, but where there are choices of x_0 such that the sequence x_n defined by setting $x_{n+1} = Tx_n$ does not converge.

23. Consider $\mathbf{M}_n(\mathbb{R})$, the space of $n \times n$ real matrices. Show that there are open neighbourhoods U and V of the identity matrix I such that if $B \in V$ then $B^4 \in U$, and for every $A \in U$ there is a unique $B \in V$ with $B^4 = A$.

24. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying a Lipschitz condition
$$|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2|.$$
Suppose that $y = g(x)$ is a solution to the differential equation
$$\frac{dy}{dx} = f(x,y)$$
with $y = y_0$ at $x = x_0$ on an interval (a,b) with $x_0 \in (a,b)$.

   (i) Show that if $g(x)$ is bounded on (a,b), then g is uniformly continuous on (a,b).

   (ii) Show that if g is uniformly continuous on (a,b), then g extends to a continuous function on $[a,b]$.

   (iii) Hence show that if $g(x)$ is bounded on (a,b), then g extends to a solution h of the differential equation on an interval (c,d) strictly including (a,b).

   (iv) Deduce that if (a,b) is a maximal interval on which we have a solution $y = g(x)$ of the differential equation, then the solution $g(x)$ is unbounded on (a,b).

25. Suppose $h : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \to \mathbb{R}$ are continuous. Show that for λ sufficiently small, the map $T : C[a,b] \to C[a,b]$ defined by $Tf(x) = \lambda \int_a^b K(x,y)f(y)dy + h(x)$ is a contraction. Deduce that, for λ sufficiently small, the (Fredholm) integral equation
$$f(x) = \lambda \int_a^b K(x,y)f(y)dy + h(x)$$
has a unique solution in $C[a,b]$.

26. What are the solutions to the differential equations
$$(i) \quad \frac{dy}{dx} = y^2, \qquad (ii) \quad \frac{dy}{dx} = 3y^{2/3}, \qquad (iii) \quad \frac{dy}{dx} = 2x(1+y^2)?$$