# ANALYSIS II EXAMPLES 2

**Michaelmas 2005** **J. M. E. Hyland**

Again the Basic Questions focus on the examinable component of the course, while with Additional Questions are for those wishing to take things further. The questions are not equally difficult; most of the hardest are among the Additional Questions.
The sheet is almost identical to last year's sheet, itself a modification of a sheet prepared by Gabriel Paternain.
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**1**. Let $\alpha\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that

$$\sup\{\|\alpha(\mathbf{x})\| : \mathbf{x} \in \mathbb{R}^n,\ \|\mathbf{x}\| \leq 1\} = \inf\{k \in \mathbb{R} : k \text{ is a Lipschitz constant for } \alpha\}\ .$$

Show that the function which sends $\alpha$ to the common value $\|\alpha\|$ of these two expressions is a norm on the vector space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ of all linear maps $\mathbb{R}^n \to \mathbb{R}^m$. [This is the *operator norm* on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.] Show also that

$$\|\alpha\| = \sup\{\|\alpha(\mathbf{x})\| : \mathbf{x} \in \mathbb{R}^n,\ \|\mathbf{x}\| = 1\} = \sup\{\|\alpha(\mathbf{x})\|/\|\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n,\ \mathbf{x} \neq \mathbf{0}\}\,.$$

**2**. Let $\ell_c$ be the space of all real sequences $(x_n)_{n=1}^{\infty}$ such that all but finitely many of the $x_n$ are zero. With the natural (pointwise) addition and scalar multiplication $\ell_c$ is a real vector space. Find two norms in $\ell_c$ which are not Lipschitz equivalent, showing explicitly that they are norms. Can you find uncountably many norms which are not Lipschitz equivalent?

**3**. Prove again the following facts about convergence of sequences in an arbitrary normed space:
(i) If $(x_n) \to x$ and $(y_n) \to y$, then $(x_n + y_n) \to x + y$.
(ii) If $(x_n) \to x$ and $\lambda \in \mathbb{R}$, then $(\lambda x_n) \to \lambda x$.
(iii) If $x_n = x$ for all $n \geq n_0$, then $(x_n) \to x$.
(iv) If $(x_n) \to x$, then any subsequence $(x_{n_i})$ of $(x_n)$ also converges to $x$.

**4**. Which of the following subsets of $\mathbb{R}^2$ are $(a)$ open, $(b)$ closed? (And why?)

(i) $\{(x, 0) : 0 \leq x \leq 1\}$. (ii) $\{(x, 0) : 0 < x < 1\}$. (iii) $\{(x, y) : y \neq 0\}$.
(iv) $\{(x, y) : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$. (v) $\{(x, y) : xy = 1\}$.

**5**. Let $E$ be a subset of $\mathbb{R}^n$ which is both open and closed. Show that $E$ is either the whole of $\mathbb{R}^n$ or the empty set. [Method: suppose for a contradiction that $x \in E$ but $y \in \mathbb{R}^n \setminus E$. Define a function $f\colon [0, 1] \to \mathbb{R}$ by setting $f(t) = 1$ if the point $tx + (1 - t)y$ belongs to $E$, and $f(t) = 0$ otherwise; now recall a suitable theorem from Analysis I.]

**6**. (i) Show that the mapping $\mathbb{R}^{2n} \to \mathbb{R}^n$ which sends a $2n$-dimensional vector

$$(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$$

to

$$(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

is continuous. Deduce that if $f$ and $g$ are continuous functions from $E \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$, then so is their (pointwise) sum $f + g$.
(ii) By considering a suitable function $\mathbb{R}^{n+1} \to \mathbb{R}^n$, give a similar proof that if $f$ is a continuous $\mathbb{R}^m$-valued function on $E \subseteq \mathbb{R}^n$, and $\lambda$ is a continuous real-valued function on $E$, then the pointwise scalar product $\lambda f$ (i.e. the function whose value at $x$ is $\lambda(x).f(x)$) is continuous on $E$.

**7**. If $A$ and $B$ are subsets of $\mathbb{R}^n$, we write $A+B$ for the set $\{a+b : a \in A, b \in B\}$. Show that if $A$ and $B$ are both closed and one of them is bounded, then $A+B$ is closed. Give an example in $\mathbb{R}^1$ to show that the boundedness condition cannot be omitted. If $A$ and $B$ are both open, is $A+B$ necessarily open? Justify your answer.

**8**. Let $f : \mathbb{R}^n \to \mathbb{R}^m$, and let $E, F$ be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Determine which of the following statements are always true and which may be false, giving a proof or a counterexample as appropriate. [N.B. For the counterexamples, it suffices to take $n = m = 1$.]

(i) If $f^{-1}(F)$ is closed whenever $F$ is closed, then $f$ is continuous.
(ii) If $f$ is continuous, then $f^{-1}(F)$ is closed whenever $F$ is closed.
(iii) If $f$ is continuous, then $f(E)$ is open whenever $E$ is open.
(iv) If $f$ is continuous, then $f(E)$ is bounded whenever $E$ is bounded.
(v) If $f(E)$ is bounded whenever $E$ is bounded, then $f$ is continuous.

**9**. In lectures we proved that if $E$ is a closed and bounded set in $\mathbb{R}^n$, then any continuous function defined on $E$ has bounded image. Prove the converse: if every continuous real-valued function on $E \subseteq \mathbb{R}^n$ is bounded, then $E$ is closed and bounded.
Now suppose that every bounded continuous real-valued function on $E \subseteq \mathbb{R}^n$ attains its bounds. Does it again follow that $E$ is closed and bounded?

**10**. Consider the vector space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ of all linear maps $\mathbb{R}^n \to \mathbb{R}^m$, equipped with the operator norm defined in Question 1.

(i) Show that if $\|\alpha\| < \varepsilon$ then all the entries in the matrix $A$ representing $\alpha$ (with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$) have absolute value less than $\varepsilon$. Conversely, show that if all entries of the matrix $A$ have absolute value less than $\epsilon$, then the norm of the linear map $\alpha$ represented by $A$ is less than $nm\varepsilon$.

(ii) Deduce that convergence for sequences of linear maps is equivalent to 'entry-wise' convergence of the representing matrices, and so that $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is complete. How else might you prove this?

(iii) If $\alpha : \mathbb{R}^n \to \mathbb{R}^m$ and $\beta : \mathbb{R}^m \to \mathbb{R}^p$ are linear maps, show that the norm of the composite $\beta \circ \alpha$ is less than or equal to the product $\|\beta\|.\|\alpha\|$.

(iv) Now specialize to the case $n = m$. Show that if $\alpha$ is an endomorphism of $\mathbb{R}^n$ satisfying $\|\alpha\| < 1$, then the sequence whose $k$th term is $\iota + \alpha + \alpha^2 + \cdots + \alpha^{k-1}$ converges (here $\iota$ denotes the identity mapping), and deduce that $\iota - \alpha$ is invertible.

(v) Deduce that if $\alpha$ is invertible then so is $\alpha - \beta$ whenever $\|\beta\| < \|\alpha^{-1}\|^{-1}$. Hence conclude that the set of invertible linear maps is open in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

**11**. Suppose that $g : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function. Show that one can define a map $G : C[0,1] \to C[0,1]$ by setting

$$(Gf)(x) = \int_0^1 g(x,t) f(t) dt$$

for $f \in C[0,1]$.

Show further that $G : C[0,1] \to C[0,1]$ is a linear map, and that $G$ is a continuous map with respect to the uniform (sup) norm. Is it continuous with respect to the $L^1$ norm? (Can you justify your answer? If not at least have a guess!)

**12**. Recall from lectures the normed space $\ell^2$. The Hilbert cube is the subset of $\ell^2$ consisting of all $(x_n)_{n=1}^{\infty} = (x_1, x_2, x_3, \cdots)$ such that for each $n$, $|x_n| \leq 1/n$. Show that the Hilbert cube is closed in $\ell^2$, and that it has the Bolzano-Weierstrass property, that is, any sequence in the Hilbert cube has a convergent subsequence in it. (So the Hilbert cube is *compact*).

## Additional Questions

**13**. Let $R[0,1]$ be the space of Riemann integrable functions on $[0,1]$.

(i) Why is $R[0,1]$ equipped with $\|f\| = \int_0^1 |f(t)|dt$ not a normed space? What could you imagine doing to remedy the situation?

(ii) Is $R[0,1]$ equipped with $\|f\| = \sup\{|f(t)| : t \in [0,1]\}$ a normed space? Is it complete?

**14**. (i) Let $C_b(\mathbb{R})$ be the space of continuous bounded functions $f : \mathbb{R} \to \mathbb{R}$ equipped with the uniform (sup) norm. Show that $C_b(\mathbb{R})$ is complete.

(ii) Let $C_0(\mathbb{R})$ be the subspace of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \to 0$ as $|x| \to \infty$. Is $C_0(\mathbb{R})$, equipped with the uniform norm, complete?

(iii) Let $C_c(\mathbb{R})$ be the subspace of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $|x|$ sufficiently large. Is $C_c(\mathbb{R})$, equipped with the uniform norm, complete?

**15**. Recall the normed space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ equipped with the operator norm. We now write the elements of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ as matrices $A$, $B$, ... .

(i) Show that in the operator norm the sum

$$\sum_{r=0}^{\infty} A^r/r!$$

converges to a matrix (and so linear map) $\exp A$.

(ii) Show that if $A$ and $B$ commute, then $\exp(A + B) = \exp A \exp B$.

(iii) What happens when $A$ and $B$ do not commute? (If you want to know more, look up the Campbell-Hausdorff formula.)

**16**. Let $T : E \to F$ be a linear map between normed spaces. Prove that the following are equivalent.

(i) $T$ is continuous.

(ii) $T$ is continuous at $\mathbf{0}$.

(iii) There is $0 < K < \infty$ such that $\|T(x)\| \leq K\|x\|$ for all $x \in E$.

In such circumstances, $T$ is a *bounded linear operator*. (What is the moral of this equivalence?)

Let $\mathcal{B}(E, F)$ be the space of bounded linear operators equipped with the usual operator norm $\|T\| = \sup\{\|T(x)\| : x \in E \text{ and } \|x\| \leq 1\}$. Show that if $F$ is complete then so is $\mathcal{B}(E, F)$.

**17**. A special case of the space $\mathcal{B}(E, F)$ of bounded linear operators gives the *dual* $E^*$ of a normed space. It is defined to be $E^* = \mathcal{B}(E, \mathbb{R})$, the space of *bounded linear functionals* with the operator norm. Now recall from lectures the normed spaces $\ell^p$, $1 \leq p \leq \infty$.

(i) Show that the dual of $\ell^1$ is isomorphic to $\ell^\infty$.

(ii) Show that the dual of $\ell^2$ is isomorphic to $\ell^2$.

Now let $c_0 = \{(x_n)_{n=1}^\infty : x_n \to 0 \text{ as } n \to \infty\}$, equipped with the sup norm. Is $c_0$ complete? Identify the dual of $c_0$. Is the dual of $\ell^\infty$ isomorphic to $\ell^1$? (Maybe you do not have enough background for the last point, but have a guess!)

**18**. Here is a different take on the relationship between $\ell^1$ and $\ell^\infty$.

(i) Let $(x_i)_{i=1}^\infty$ be such that, for all $(y_i)_{i=1}^\infty \in \ell^\infty$, $\sum_i x_i y_i$ converges. Show that $(x_i)_{i=1}^\infty \in \ell^1$.

(ii) Let $(y_i)_{i=1}^\infty$ be such that, for all $(x_i)_{i=1}^\infty \in \ell^1$, $\sum_i x_i y_i$ converges. Show that $(x_i)_{i=1}^\infty \in \ell^\infty$.

(iii) Suppose that $\mathbf{x}^{(n)} = (x_i^{(n)})_{i=1}^\infty$ is a sequence in $\ell^1$ such that, for all $(y_i)_{i=1}^\infty \in \ell^\infty$, $\sum_i x_i^{(n)} y_i \to 0$ as $n \to \infty$. Does $\mathbf{x}^{(n)} \to \mathbf{0}$ in $\ell^1$?

(iv) Suppose that $\mathbf{y}^{(n)} = (y_i^{(n)})_{i=1}^\infty$ is a sequence in $\ell^\infty$ such that, for all $(x_i)_{i=1}^\infty \in \ell^1$, $\sum_i x_i y_i^{(n)} \to 0$ as $n \to \infty$. Does $\mathbf{y}^{(n)} \to \mathbf{0}$ in $\ell^\infty$?

**19**. Suppose that $E$ is a normed space in which the unit ball $\{x : \|x\| \leq 1\}$ is compact (in the sense that the Bolzano-Weierstrass Theorem holds for it). Show that $E$ is finite dimensional.

**20**. Let $E$ be a normed space. Can there exist bounded linear operators $S, T : E \to E$ such that $S \circ T - T \circ S = I$ where $I : E \to E$ is the identity?