ANALYSIS II EXAMPLES 4

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These Basic Questions cover the last quarter of the course as it now is. There is some overlap with questions from Gabriel Paternain's sheets last year, but less than before: compactness as a general phenomenon is now treated in Metric and Topological Spaces. There are too many Additional Questions: but they should not do you any harm. Comments and corrections are welcome as ever and may be e-mailed to me at `m.hyland@dpmms.cam.ac.uk`.

Basic Questions

1. For each of the following sets X, determine whether the given function d defines a metric on X:

(i) $X = \mathbb{R}^n$, $d(x, y) = \min\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.

(ii) $X = \mathbb{Z}$, $d(x, x) = 0$ for all x, otherwise $d(x, y) = 2^n$ if $x - y = 2^n a$ where a is odd.

(iii) $X = \mathbb{Q}$, $d(x, x) = 0$ for all x, otherwise $d(x, y) = 3^{-n}$ if $x - y = 3^n a/b$ where a, b are prime to 3 (and n may be positive, negative or zero).

(iv) $X = \{\text{functions } \mathbb{N} \to \mathbb{N}\}$, $d(f, f) = 0$, otherwise $d(f, g) = 2^{-n}$ for the least n such that $f(n) \neq g(n)$.

(v) $X = \mathbb{C}$, $d(z, w) = |z - w|$ if z and w are on the same straight line through 0, otherwise $d(z, w) = |z| + |w|$.

2. Let (X, d) be a metric space. Show that

$$d_1(x, y) = \min(1, d(x, y)) \quad \text{and} \quad d_2(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

are metrics on X topologically equivalent to d. Are the metrics d, d_1 and d_2 uniformly equivalent? Are they Lipschitz equivalent?

3. (i) Suppose that $A \subseteq X$ is a subset in a metric space (X, d). Show that

$$d(x, A) = \inf_{a \in A} d(x, a)$$

defines a continuous real-valued function on X.

(ii) Suppose that A and B are disjoint closed sets in a metric space (X, d). Show that there exist disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

(iii) Give an example of disjoint closed sets A and B in $\mathbb{R}^2$ with points $a_n \in A$ and $b_n \in B$ such that $d(a_n, b_n) \to 0$ as $n \to \infty$. (So disjoint closed sets need not be a finite distance apart.)

4. Suppose that (X, d_X) is a metric space. For $Y \subseteq X$ let (Y, d_Y) be the induced metric subspace.

(i) Show that U is open in (Y, d_Y) if and only if $U = Y \cap V$ for some V open in (X, d_X).

(ii Show that A is closed in (Y, d_Y) if and only if $A = Y \cap B$ for some B closed in (X, d_X).

5. (i) Show that no two of the intervals $[0, 1]$, $[0, 1)$ and $(0, 1)$ are homeomorphic.

(ii) Is the unit circle $\{(x, y) : x^2 + y^2 = 1\}$ homeomorphic to any interval?

6. (i) Show that the space of real sequences $\mathbf{a} = (a_n)_{n=1}^{\infty}$ with all but finitely many of the a_n are zero is not complete in the norm defined by $\|\mathbf{a}\|_1 = \sum_{n=1}^{\infty} |a_n|$. Is there an obvious 'completion'?

(ii) Show that $d(f, g) = \int_a^b |f(x) - g(x)|\, dx$ is a metric on $\mathcal{C}[a, b]$ the space of continuous functions on $[a, b]$. Is $(\mathcal{C}[a, b], d)$ complete?

7. Show that $x = \cos x$ has a unique solution. Use a reasonable pocket calculator to find the solution to some decimal places. (This should take no time. Remember to work in radians!)

8. Consider the map $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x^3 - 3x^2 + 3x$. For which initial values $x_0 \in \mathbb{R}$ does the sequence of iterates $x_n = T^n(x_0)$ converge and to what value? (Clearly it will be helpful to sketch a graph; but you should give proofs!)

9. Suppose that (X, d) is a (not necessarily complete) metric space and that $T : X \to X$ is a contraction.

(i) Show that if T has a fixed point, then it is unique.

(ii) Show that for any choice of x_0, if the sequence x_n defined by setting $x_{n+1} = Tx_n$ converges, then it converges to a fixed point of T.

(iii) Show that if T has a fixed point, then for any choice of x_0, the sequence x_n defined by setting $x_{n+1} = Tx_n$ converges to the fixed point.

(iv) Give an example of a non-empty metric space (X, d) and contraction T with no fixed point.

10. [Tripos IB 96401(b), modified] (i) Suppose that (X, d) is a nonempty complete metric space, and $f : X \to X$ a continuous map such that, for any $x, y \in X$, the sum $\sum_{n=1}^{\infty} d(f^n(x), f^n(y))$ converges. Show that f has a unique fixed point.

(ii) By considering the function $x \mapsto \max\{x - 1, 0\}$ on the interval $[0, \infty) \subseteq \mathbb{R}$, show that a function satisfying the hypotheses of (i) need not be a contraction mapping.

(iii) Let ϕ be a continuous real-valued function on $\mathbb{R} \times [a, b]$ which satisfies the Lipschitz condition

$$|\phi(x, t) - \phi(y, t)| \le M\,|x - y| \ , \text{ for all } x, y \in \mathbb{R} \text{ and } t \in [a, b],$$

and let $g \in \mathcal{C}[a, b]$. Define $F : \mathcal{C}[a, b] \to \mathcal{C}[a, b]$ by $F(h)(t) = g(t) + \int_a^t \phi(h(s), s)\, ds$. Show by induction that

$$|F^n(h)(t) - F^n(k)(t)| \le \frac{1}{n!} M^n (t - a)^n \, \|h - k\|_\infty \ ,$$

for all $h, k \in \mathcal{C}[a, b]$ and $a \le t \le b$, and deduce that F has a unique fixed point.

(iv) In the original 1996 Tripos question from which this question was adapted, the word 'continuous' in the second line of part (i) was accidentally omitted. Give a counterexample to the result which the 1996 IB students were asked to prove.

11. [Tripos IB 95401(b)] For which a and b, with $a \le 0 \le b$, is the mapping $T : \mathcal{C}[a, b] \to \mathcal{C}[a, b]$ defined by

$$T(f)(x) = 1 + \int_0^x 2t\, f(t)\, dt$$

a contraction? Deduce that the differential equation

$$\frac{dy}{dx} = 2xy \ , \quad \text{with } y = 1 \text{ when } x = 0 \ ,$$

has a unique solution in some interval containing 0. In what interval can the differential equation be solved?

12. For fixed $B \in \mathbf{M}_n(\mathbb{R})$ define $f_B : \mathbf{M}_n \to \mathbf{M}_n$ by

$$f_B(X) = X - \exp(X) + B \,.$$

Show directly that if $\|X\| \le K$ and $\|Y\| \le K$, then

$$\|f_B(X) - f_B(Y)\| \le \|X - Y\|(e^K - 1) \,.$$

Show that if $\|I - B\| \le 1/5$ then f_B maps the ball $\{X : \|X\| \le 1/3\}$ into itself, and deduce that the equation $\exp(X) = B$ has a solution for $\|X\| \le 1/3$.

Additional Questions

13. A metric d on a set X is called an *ultrametric* if it satisfies the following stronger form of the triangle inequality:

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \quad \text{for all } x,y,z \in X\ .$$

Which of the metrics in question 1 are ultrametrics? Show that in an ultrametric space 'every triangle is isosceles' (that is, at least two of $d(x,y)$, $d(y,z)$ and $d(x,z)$ must be equal), and deduce that every open ball in an ultrametric space is a closed set. Does it follow that every open set must be closed?

14. There is a persistent 'urban myth' about the mathematics research student who spent three years writing a thesis about properties of 'antimetric spaces', where an *antimetric* on a set X is a function $d\colon X \times X \to \mathbb{R}$ satisfying the same axioms as a metric except that the triangle inequality is reversed (i.e. $d(x,z) \geq d(x,y) + d(y,z)$ for all x,y,z). Why would such a thesis be unlikely to be considered worth a Ph.D.?

15. Show that the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} x & \text{if } y \text{ is irrational,} \\ 2x & \text{if } y \text{ is rational,} \end{cases}$$

takes open sets to open sets, but is not continuous.

16. Suppose (X, d_X) and $Y, d_Y)$ are metric spaces and $f : X \to Y$ is a map between them.

(i) Let $\{U_i\ :\ i \in I\}$ be a family of open subsets of X with $X = \bigcup\{U_i\ :\ i \in I\}$. Show that f is continuous if and only if all the restrictions $f_i : U_i \to Y$ of f to U_i are continuous.

(ii) Let A and B be closed subsets of X with $A \cup B = X$. Show that f is continuous if and only if all the restrictions $f_A : A \to Y$ and $f_B : B \to Y$ of f to A and B respectively are continuous.

17. (i) Show that $\mathbf{GL}_n(\mathbb{R})$, the collection (group) of all invertible matrices, is an open subset of the space $\mathbf{M}_n(\mathbb{R})$ of all matrices.
[*Hint: Show that* $\mathbf{GL}_n(\mathbb{R})$ *is the inverse image under a continuous map of some (simple) open set.*]

(ii) Show that $\mathbf{O}_n(\mathbb{R})$, the collection (group) of all orthogonal matrices is a closed subset of the space $\mathbf{M}_n(\mathbb{R})$ of all matrices.
[*Hint: Show that* $\mathbf{O}_n(\mathbb{R})$ *is the inverse image under a continuous map of some (simple) closed set.*]

18. Suppose that $f : X \to Y$ is a continuous map. Consider the graph $G_f \subseteq X \times Y$ of f defined by

$$G_f = \{(x,y)\ :\ f(x) = y\}\,.$$

(i) Show that G_f is closed in $X \times Y$ (equipped with a product metric).

(ii) Show that the subspace G_f of $X \times Y$ is homeomorphic to X.

19. Let E be a subset of $\mathbb{R}^n$ (or, if you prefer, of an arbitrary metric space). We define the *closure* $\overline{E}$ of E to be the set of all points which can occur as limits of sequences of points of E, and the *interior* E° of E to be the set

$$\{x \in \mathbb{R}^n : (\exists\epsilon > 0)(B(x,\epsilon) \subseteq E)\}\ .$$

(i) Show that $\overline{E}$ is closed, and that it is the smallest closed set containing E.

(ii) Show that E° is open, and that it is the largest open set contained in E.

(iii) Show that $\overline{\mathbb{R}^n \setminus E} = \mathbb{R}^n \setminus E^\circ$.

(iv) By considering the inclusion relations which must hold amongst the sets

$$\ldots, \overline{(\overline{E})^\circ}, (\overline{E})^\circ, \overline{E}, E, E^\circ, \overline{E^\circ}, \ldots$$

show that starting from a given E, it is not possible to produce more than seven distinct sets by repeated application of the operators $\overline{(-)}$ and $(-)^\circ$.

(v) Find an example of a set in $\mathbb{R}^1$ which does give rise to seven distinct sets in this way.

20. [Tripos IB 93301(b)] Let (X, d) be a metric space without isolated points (i.e. such that $\{x\}$ is not open for any $x \in X$), and $(x_n)_{n\geq 0}$ a sequence of points of X. Show that it is possible to find a sequence of points y_n of X and positive real numbers r_n such that $r_n \to 0$, $d(x_n, y_n) > r_n$ and

$$B(y_n, r_n) \subseteq B(y_{n-1}, r_{n-1})$$

for each $n > 0$. Deduce that a nonempty complete metric space without isolated points has uncountably many points. [This is a direct generalization of a proof of the uncountability of $\mathbb{R}$.]

21. Suppose that (X, d) is a complete metric space. Let $U \subset X$ be a proper open subset with complement $A = X - U$. Show the following.

(i) $d(x, A) = \inf\{d(x, a) : a \in A\}$ is a continuous real-valued function on X.

(ii) $f(x) = (d(x, A))^{-1}$ is a continuous real-valued function on U.

(iii) $\bar{d}(x, y) = d(x, y) + |f(x) - f(y)|$ is a metric on U.

(iv) d and $\bar{d}$ are equivalent metrics on U, but are not in general uniformly equivalent.

(v) $(U, \bar{d})$ is a complete metric space.

22. Is $T : C[0, 1] \to C[0, 1]\,;\ Tf(x) = \int_0^x f(t^2)dt$ a contraction mapping? Show that T has a unique fixed point.

23. Suppose (X, d) is a metric space and $T : X \to X$ satisfies $d(Tx, Ty) < d(x, y)$ for all $x \neq y$ in X.

(i) Show that if T has a fixed point, then it is unique.

(ii) Show that for any choice of x_0, if the sequence x_n defined by setting $x_{n+1} = Tx_n$ converges, then it converges to a fixed point of T.

(iii) Give an example of a metric space (X, d) and map $T : X \to X$, satisfying $d(Tx, Ty) < d(x, y)$ for all $x \neq y$ in X, and with a fixed point, but where there are choices of x_0 such that the sequence x_n defined by setting $x_{n+1} = Tx_n$ does not converge.

24. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function satisfying a Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|\,.$$

Suppose that $y = g(x)$ is a solution to the differential equation

$$\frac{dy}{dx} = f(x, y)$$

with $y = y_0$ at $x = x_0$ on an interval (a, b) with $x_0 \in (a, b)$.

(i) Show that if $g(x)$ is bounded on (a, b), then g is uniformly continuous on (a, b).

(ii) Show that if g is uniformly continuous on (a, b), then g extends to a continuous function on $[a, b]$.

(iii) Hence show that if $g(x)$ is bounded on (a, b), then g extends to a solution h of the differential equation on an interval (c, d) strictly including (a, b).

(iv) Deduce that if (a, b) is a maximal interval on which we have a solution $y = g(x)$ of the differential equation, then the solution $g(x)$ is unbounded on (a, b).

25. Suppose $h : [a, b] \to \mathbb{R}$ and $K : [a, b]^2 \to \mathbb{R}$ are continuous. Show that for λ sufficiently small, the map $T : C[a, b] \to C[a, b]$ defined by $Tf(x) = \lambda \int_a^b K(x, y)f(y)dy + h(x)$ is a contraction. Deduce that, for λ sufficiently small, the (Fredholm) integral equation

$$f(x) = \lambda \int_a^b K(x, y)f(y)dy + h(x)$$

has a unique solution in $C[a, b]$.

26. What are the solutions to the differential equations

$$(i)\quad \frac{dy}{dx} = y^2\,, \qquad (ii)\quad \frac{dy}{dx} = 3y^{2/3}\,, \qquad (iii)\quad \frac{dy}{dx} = 2x(1 + y^2)\,?$$