ANALYSIS II EXAMPLES 3

Michaelmas 2004 **J. M. E. Hyland**

This sheet contains Basic Questions covering the examinable material from the course, together with miscellaneous Additional Questions. The questions are not equally difficult. Question 16 may be hard, but merits particular attention once you have done the Basic Questions. The sheet is a modification of Gabriel Paternain's sheet from last year. Comments and corrections are welcome and should be sent to `m.hyland@dpmms.cam.ac.uk`.

Basic Questions

1. Consider the mapping $\Omega\colon \mathbb{R}^6 \to \mathbb{R}^3$ defined by $\Omega(\mathbf{x}, \mathbf{y}) = \mathbf{x} \wedge \mathbf{y}$ (i.e. the usual 'cross product' of three-dimensional vectors). Prove directly from the definition that Ω is differentiable everywhere. (Of what is this a special case?) Express the derivative at $(\mathbf{x}, \mathbf{y})$ first as a linear map and then as a Jacobian matrix of partial derivatives.

2. Let $f(x, y) = x^2y/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$. Show that f is continuous at $(0, 0)$, and that it has directional derivatives in all directions there (i.e., for any fixed α, the function $t \mapsto f(t \cos\alpha, t \sin\alpha)$ is differentiable at $t = 0$). Is f differentiable at $(0, 0)$?

3. At which points of $\mathbb{R}^2$ are the following functions $\mathbb{R}^2 \to \mathbb{R}$ differentiable?
(i) $f(x, y) = xy\,|x - y|$.
(ii) $f(x, y) = xy/\sqrt{(x^2 + y^2)} \quad ((x, y) \neq (0, 0))$, $f(0, 0) = 0$.
(iii) $f(x, y) = xy \sin 1/x \quad (x \neq 0)$, $f(0, y) = 0$.

4. (i) Let V be a finite dimensional real vector space equipped with an inner product $\langle -, - \rangle$, and let $\| - \|$ be the norm derived from this inner product (i.e. $\|x\| = \sqrt{\langle x, x \rangle}$). Show that the function $V \to \mathbb{R}$ sending x to $\|x\|$ is differentiable at all nonzero $x \in V$. [Hint: can you show that $x \mapsto \|x\|^2$ is differentiable?]

(ii) At which points in $\mathbb{R}^2$ are the functions $\| - \|_1$ and $\| - \|_\infty$ differentiable? (The shapes of the unit balls give a clue to where differentiability can be expected to fail.)

5. Prove *Euler's Theorem*: if $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and satisfies the identity

$$f(tx, ty) = t^n f(x, y)$$

(i.e. f is *homogeneous of degree* n), then f satisfies the partial differential equation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf\,.$$

6. We work in $\mathbb{R}^3$ with the usual inner product. Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x) = \frac{x}{\|x\|} \quad \text{for } x \neq 0$$

and $f(0) = 0$. Show that f is differentiable except at 0 and

$$Df(x)(h) = \frac{h}{\|x\|} - \langle x, h \rangle \frac{x}{\|x\|^3}.$$

Verify that $Df(x)(h)$ is orthogonal to x and explain geometrically why this is the case.

7. (i) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $D_1 f = \partial f/\partial x$ is continuous in a neighbourhood of (a, b), and $D_2 f = \partial f/\partial y$ exists at (a, b). Show that f is differentiable at (a, b).
(ii) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $D_1 f = \partial f/\partial x$ exists and is bounded near (a, b), and that for a fixed, $f(a, y)$ is continuous as a function of y. Show that f is continuous at (a, b).

8. Put $f(0, 0) = 0$, and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x, y) \neq (0, 0)$. Prove that
(a) f, $D_1 f$, $D_2 f$ are continuous in $\mathbb{R}^2$;
(b) $D_{12} f$ and $D_{21} f$ exist at every point in $\mathbb{R}^2$;
(c) $D_{12} f(0, 0) = 1$ and $D_{21} f(0, 0) = -1$.
Where are $D_{12} f$ and $D_{21} f$ continuous?

9. [Tripos IB 98210, modified] Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be the space of linear maps $\mathbb{R}^n \to \mathbb{R}^n$. This is most naturally equipped with the operator norm (cf. Examples sheet 2), but can be identified with $\mathbb{R}^{n^2}$. Consider the function $f : \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ defined by $f(\alpha) = \alpha^2$. Show that f is differentiable everywhere in V. Is it true that $f'(\alpha) = 2\alpha$? If not, what is the derivative of f at α?

Now let $\mathbf{GL}_n(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be the open subset consisting of invertible endomorphisms, and let $g : \mathbf{GL}_n(\mathbb{R}) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be defined by $g(\alpha) = \alpha^{-1}$. Show that g is differentiable at ι (the identity mapping), and that its derivative at ι is the linear mapping $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ which sends β to $-\beta$. Suppose now that α and $\alpha + \gamma$ are both in U; verify that

$$(\alpha + \gamma)^{-1} - \alpha^{-1} = [(\iota + \alpha^{-1}\gamma)^{-1} - \iota]\alpha^{-1} \ .$$

Hence, or otherwise, show that g is differentiable at $\alpha \in \mathbf{GL}_n(\mathbb{R})$, and find its derivative there.

10. To vary the setting, let us take $\mathbf{M}_n = \mathbf{M}_n(\mathbb{R})$ to be the space of $n \times n$ real matrices. It can be identified with $\mathbb{R}^{n^2}$.

(i) Let $h : \mathbf{M}_n \to \mathbf{M}_n$ be defined by $h(A) = A^3$. Find the repeated derivatives $h^{(r)}(A)$ of h at A. Verify that the Taylor's Theorem expansion of $\gamma(A + H)$ about A is what it ought to be.

(ii) (This assumes you did the last part of the previous question!) Let $\mathbf{GL}_n(\mathbb{R}) \subseteq \mathbf{M}_n$ be the open subset consisting of invertible $n \times n$ real matrices. Again we let $g : \mathbf{GL}_n(\mathbb{R}) \to \mathbf{M}_n$ be defined by $g(A) = A^{-1}$. Find the repeated derivatives $g^{(r)}(A)$ of g at $A \in \mathbf{GL}_n(\mathbb{R})$. What is the Taylor expansion of $g(A + H)$ about A. Is it what you expected?

11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $g(x) = f(x, c - x)$ where c is a constant. Show that $g : \mathbb{R} \to \mathbb{R}$ is differentiable and find its derivative
(i) directly from the definition of differentiability
and also
(ii) by using the chain rule.
Deduce that if $D_1 f = D_2 f$ holds throughout $\mathbb{R}^2$ then $f(x, y) = h(x + y)$ for some differentiable function h.

12. If f is a real function defined in a convex open set $E \subset \mathbb{R}^n$ such that $D_1 f(x) = 0$ for all $x \in E$, prove that $f(x)$ only depends on $x_2, \ldots, x_n$. What can you say if E is not convex? [Recall that E is said to be convex if $\lambda x + (1 - \lambda)y \in E$ whenever $x \in E$, $y \in E$ and $\lambda \in (0, 1)$.]

Additional Questions

13. (i) Show that $f(x,y)=\sqrt{|xy|}$ is continuous at $(0,0)$, that both partial derivatives exist, but that no other directional derivatives exist there.

(ii) Show that

$$f(x,y)=\begin{cases} x/y & \text{if } y\neq 0,\\ 0 & \text{otherwise,}\end{cases}$$

has both partial derivatives, no other directional derivatives, and is discontinuous at $(0,0)$.

(iii) Consider the function

$$f(x,y)=\begin{cases} x^2y^2(x^3-y^2)^{-1} & \text{if } x^3\neq y^2,\\ 0 & \text{otherwise.}\end{cases}$$

Show that f has all directional derivatives, and that the directional derivative in the direction (u,v) is $u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}$ as one expects. Show however that f is not continuous at $(0,0)$.

(iv) Show that the function

$$f(x,y)=\begin{cases} x^2+y^2 & \text{if } x \text{ and } y \text{ are rational,}\\ 0 & \text{otherwise,}\end{cases}$$

is continuous just at the point $(0,0)$; and that in fact f is differentiable there.

(v) Show that the function

$$f(x,y)=\begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y)\neq(0,0),\\ 0 & \text{otherwise,}\end{cases}$$

is differentiable, but that its partial derivatives are not continuous at $(0,0)$.

14. Compute the partial derivatives

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) \quad\text{and}\quad \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right).$$

Now consider the function

$$f(x,y)=\begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{otherwise.}\end{cases}$$

Compute

$$\int_0^1\left(\int_0^1 f(x,y)dx\right)dy \quad\text{and}\quad \int_0^1\left(\int_0^1 f(x,y)dy\right)dx\,.$$

Explain this disagreeable result.

15. Let $U\subset\mathbb{R}^2$ be an open set that contains the rectangle $[a,b]\times[c,d]$. Suppose that $g:U\to\mathbb{R}$ is continuous and that the partial derivative D_2g exists and is continuous. Set

$$G(y)=\int_a^b g(x,y)\,dx.$$

Show that G is differentiable on (c,d) with derivative

$$G'(y)=\int_a^b D_2g(x,y)\,dx.$$

Show further that

$$H(y)=\int_a^y g(x,y)\,dx$$

is differentiable. What is its derivative $H'(y)$?

16. Let $\mathbf{M}_n(\mathbb{R})$ denote the vector space of all $(n \times n)$ real matrices, which may be equipped with any suitable norm.

(i) By considering $\det(I+H)$ as a polynomial in the entries of H, show that the familiar function $\det\colon \mathbf{M}_n(\mathbb{R}) \to \mathbb{R}$ is differentiable at the identity matrix I and that its derivative there is the linear map $H \mapsto \operatorname{tr} H$.

(ii) Hence show that det is differentiable at any invertible matrix A, and that its derivative at A is the linear map $H \mapsto \det A \operatorname{tr}(A^{-1}H)$.

(iii) (Note that as the set $\mathbf{GL}_n(\mathbb{R}) \subseteq \mathbf{M}_n$ is open, all matrices sufficiently close to the identity matrix are invertible.) Calculate the second derivative of det at I as a bilinear map $\mathbf{M}_n(\mathbb{R}) \times \mathbf{M}_n(\mathbb{R}) \to \mathbb{R}$, and verify that it is symmetric.