ANALYSIS II EXAMPLES 1

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This sheet contains Basic Questions, which focus on the examinable component of the course, together with Additional Questions for those wishing to take things further. The questions are not all equally difficult; I have tried to ensure that the hardest appear amongst the Additional Questions. The sheet is a modification of Gabriel Paternain's sheet from last year. I welcome comments and corrections which can be sent to `m.hyland@dpmms.cam.ac.uk`.

Basic Questions

1. Define $f_n : [0, 2] \to \mathbb{R}$ by

$$\begin{aligned} f_n(x) &= 1 - n|x - n^{-1}| \qquad && \text{for } |x - n^{-1}| \le n^{-1}, \\ f_n(x) &= 0 && \text{otherwise.} \end{aligned}$$

Show that the f_n are continuous and sketch their graphs. Show that f_n converges pointwise on $[0, 2]$ to the zero function but not uniformly.

2. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous. Show that the sequence $x^n f(x)$ is uniformly convergent on $[0, 1]$ if and only if $f(1) = 0$.

3. Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}\,.$$

(i) Show that f_n is uniformly convergent on $(-\infty, \infty)$.
(ii) Is f'_n uniformly convergent on $[0, 1]$?
(iii) What are $\lim_{n\to\infty} f'_n(x)$ and $(\lim_{n\to\infty} f_n)'(x)$?

4. Let f and g be uniformly continuous real-valued functions on a set E.
(i) Show that the (pointwise) sum $f + g$ is uniformly continuous on E, as also is λf for any real constant λ.
(ii) Is the product fg necessarily uniformly continuous on E? Give a proof or counter-example as appropriate.

5. Which of the following functions f are (a) uniformly continuous, (b) bounded on $[0, \infty)$?
(i) $f(x) = \sin x^2$.
(ii) $f(x) = \inf\{|x - n^2| : n \in \mathbb{N}\}$.
(iii) $f(x) = (\sin x^3)/(x + 1)$.

6. (i) Show that if (f_n) is a sequence of uniformly continuous functions on $\mathbb{R}$, and $f_n \to f$ uniformly on $\mathbb{R}$, then f is uniformly continuous.
(ii) Give an example of a sequence of uniformly continuous functions f_n on $\mathbb{R}$, such that f_n converges pointwise to a continuous function f, but f is not uniformly continuous. [Hint: choose the limit function f first, and then take the f_n to be a sequence of 'approximations' to it.]

7. Suppose that f is continuous on $[0, \infty)$ and that $f(x)$ tends to a (finite) limit as $x \to \infty$. Is f necessarily uniformly continuous on $[0, \infty)$? Give a proof or a counterexample as appropriate.

8. Consider the functions $f_n\colon [0,1] \to \mathbb{R}$ defined by $f_n(x) = n^p x \exp(-n^q x)$ where p, q are positive constants.

(i) Show that f_n converges pointwise on $[0,1]$, for any p and q.

(ii) Show that if $p < q$ then f_n converges uniformly on $[0,1]$.

(iii) Show that if $p \geq q$ then f_n does not converge uniformly on $[0,1]$. Does f_n converge uniformly on $[0, 1-\epsilon]$? Does f_n converge uniformly on $[\epsilon, 1]$? [Here $0 < \epsilon < 1$; you should justify your answers.]

9. Let $f_n(x) = n^\alpha x^n (1-x)$, where α is a real constant.

(i) For which values of α does $f_n(x) \to 0$ pointwise on $[0,1]$?

(ii) For which values of α does $f_n(x) \to 0$ uniformly on $[0,1]$?

(iii) For which values of α does $\int_0^1 f_n(x)\,dx \to 0$?

(iv) For which values of α does $f_n'(x) \to 0$ pointwise on $[0,1]$?

(v) For which values of α does $f_n'(x) \to 0$ uniformly on $[0,1]$?

10. Consider the sequence of functions $f_n\colon (\mathbb{R} \setminus \mathbb{Z}) \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{m=0}^{n} (x-m)^{-2} .$$

(i) Show that f_n converges pointwise on $\mathbb{R} \setminus \mathbb{Z}$ to a function f.

(ii) Show that f_n does not converge uniformly on $\mathbb{R} \setminus \mathbb{Z}$.

(iii) Why can we nevertheless conclude that the limit function f is continuous, and indeed differentiable, on $\mathbb{R} \setminus \mathbb{Z}$?

11. Suppose f_n is a sequence of continuous functions from a bounded closed interval $[a,b]$ to $\mathbb{R}$, and that f_n converges pointwise to a continuous function f.

(i) If f_n converges uniformly to f, and (x_m) is a sequence of points of $[a,b]$ converging to a limit x, show that $f_n(x_n) \to f(x)$. [Careful — this is not quite as easy as it looks!]

(ii) If f_n does **not** converge uniformly, show that we can find a convergent sequence $x_n \to x$ in $[a,b]$ such that $f_n(x_n)$ does not converge to $f(x)$. [Hint: Bolzano–Weierstrass.]

12. (i) Suppose f is defined and differentiable on a (bounded or unbounded) interval $E \subseteq \mathbb{R}$, and that its derivative f' is bounded on E. Use the Mean Value Theorem to show that f is uniformly continuous on E.

(ii) Give an example of a function f which is (uniformly) continuous on $[0,1]$, and differentiable at every point of $[0,1]$ (here we interpret $f'(0)$ as the 'one-sided derivative' $\lim_{h \to 0^+}((f(h) - f(0))/h)$, and similarly for $f'(1)$), but such that f' is unbounded on $[0,1]$. [Hint: last year you probably saw an example of an everywhere differentiable function whose derivative is discontinuous; you will need to 'tweak' it slightly.]

Additional Questions

13. Let f be a bounded function defined on a set $E \subseteq \mathbb{R}$, and for each positive integer n let g_n be the function defined on E by

$$g_n(x) = \sup\{|f(y) - f(x)| : y \in E, |y - x| < 1/n\} .$$

Show that f is uniformly continuous on E if and only if $g_n \to 0$ uniformly on E as $n \to \infty$.

14. Show that the series

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$$

converges for $s > 1$, and is uniformly convergent on $[1+\varepsilon, \infty)$ for any $\varepsilon > 0$.

Show that ζ is differentiable on $(1, \infty)$. (First think what its derivative ought to be!)

15. (Dirichlet's Test) Let f_n and g_n be real-valued functions on the interval $[a,b]$. Suppose that for $x \in [a,b]$, $|\sum_0^N f_n(x)| \le K$, where K is constant, for all N; and suppose that $g_n(x)$ is monotonic for each $x \in [a,b]$ with $g_n \to 0$ uniformly on $[a,b]$. Show that the sum $\sum_0^\infty f_n(x)g_n(x)$ is uniformly convergent on $[a,b]$.

16. (i) (Abel's Test) Let f_n and g_n be real-valued functions on $[a,b]$. Suppose that $\sum_0^\infty f_n(x)$ is uniformly convergent on $[a,b]$, that each $g_n(x)$ is bounded on $[a,b]$, and that $g_n(x) \ge g_{n+1}(x) \ge 0$ for all $x \in [a,b]$. Show that the sum $\sum_0^\infty f_n(x)g_n(x)$ is uniformly convergent on $[a,b]$.

(ii) Deduce that if $\sum_0^\infty a_n$ is convergent, then $\sum_0^\infty a_n x^n$ is uniformly convergent on $[0,1]$. (But note that $\sum_0^\infty a_n x^n$ need not be convergent at -1; you almost certainly know a counterexample!)

17. Suppose that g_n are continuous functions with $g_n(x) \ge g_{n+1}(x)$ for all $x \in \mathbb{R}$, and with $g_n \to 0$ uniformly in $\mathbb{R}$.

(i) Show that both $\sum_{n=0}^\infty g_n(x)\cos nx$ and $\sum_{n=0}^\infty g_n(x)\sin nx$ converge uniformly on any interval of the form $[\delta, 2\pi-\delta]$, where $\delta > 0$.

(ii) Give an example to show that we do not necessarily have convergence uniformly on $[0,2\pi]$.

18. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions converging pointwise to a continuous function $f : [0,1] \to \mathbb{R}$ on the unit interval $[0,1]$. Suppose that $f_n(x)$ is a decreasing sequence for each $x \in [0,1]$. Show that $f_n \to f$ uniformly on $[0,1]$.

19. Define $\varphi(x) = |x|$ for $x \in [-1,1]$ and extend the definition of $\varphi(x)$ to all real x by requiring that

$$\varphi(x+2) = \varphi(x).$$

(i) Show that $|\varphi(s) - \varphi(t)| \le |s-t|$ for all s and t.

(ii) Define $f(x) = \sum_{n=0}^\infty \left(\frac{3}{4}\right)^n \varphi(4^n x)$. Prove that f is well defined and continuous.

(iii) Fix a real number x and positive integer m. Put $\delta_m = \pm\frac{1}{2}\,4^{-m}$ where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x+\delta_m)$. Prove that

$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| \ge \frac{1}{2}(3^m + 1).$$

Conclude that f is not differentiable at x. Hence there exists a real continuous function on the real line which is nowhere differentiable.

20. *A space-filling curve (Exercise 14, Chapter 7 of Rudin's book).* Let f be a continuous real function on $\mathbb{R}$ with the following properties: $0 \le f(t) \le 1$, $f(t+2) = f(t)$ for every t, and

$$f(t) = \begin{cases} 0 & \text{for } t \in [0,1/3]; \\ 1 & \text{for } t \in [2/3,1]. \end{cases}$$

Put $\Phi(t) = (x(t), y(t))$, where

$$x(t) = \sum_{n=1}^\infty 2^{-n} f(3^{2n-1}t), \qquad y(t) = \sum_{n=1}^\infty 2^{-n} f(3^{2n}t).$$

Prove that Φ is continuous and that Φ maps $I = [0,1]$ onto the unit square $I^2 \subset \mathbb{R}^2$. In fact, show that Φ maps the Cantor set onto I^2.

Hint: Each $(x_0, y_0) \in I^2$ has the form

$$x_0 = \sum_{n=1}^\infty 2^{-n} a_{2n-1}, \qquad y_0 = \sum_{n=1}^\infty 2^{-n} a_{2n}$$

where each a_i is 0 or 1. If

$$t_0 = \sum_{i=1}^\infty 3^{-i-1}(2a_i)$$

show that $f(3^k t_0) = a_k$, and hence that $x(t_0) = x_0$, $y(t_0) = y_0$.