IA Probability Examples Sheet 4, Lent 2008 W. T. G.

1. A shot is fired at a circular target. The vertical and horizontal coordinates of the point of impact (taking the centre of the target as the origin) are independent standard normal random variables.

(i) Show that the distance from the centre to the point of impact has density function $re^{-r^2/2}$ for $r \geq 0$.

(ii) Show that the mean of this distance is $\sqrt{\pi/2}$, that the median is $\sqrt{\log 4}$, and that the mode is 1.

2. Let X and Y be independent normal random variables with mean 0 and variance 1. For each θ let X_θ be the random variable $X\cos\theta + Y\sin\theta$. What is the covariance of X_θ and X_ϕ?

3. Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ orthogonal matrix. (This means that $AA^T = I$.) Let $X_1, \ldots, X_n$ be independent normal random variables with mean 0 and variance 1. For each i let $Y_i = a_{i1}X_1 + \ldots + a_{in}X_n$. Prove that $Y_1, \ldots, Y_n$ are independent normal random variables with mean 0 and variance 1.

4. A radioactive source emits particles in a random direction (with all directions being equally likely). It is held at a distance d from a vertical infinite plane photographic plate.

(i) Show that, given that the particle hits the plate, the horizontal coordinate of its point of impact (with the point nearest the source taken as the origin) has density function $d/\pi(d^2 + x^2)$. [This distribution is known as the *Cauchy distribution*.

(ii) Can you compute the mean of this distribution?

5. Suppose that $X_1, \ldots, X_{2n+1}$ are i.i.d. random variables that form a random sample from the $U(0,1)$ distribution. Suppose that the values are arranged in increasing order as $Y_1 \leq Y_2 \leq \ldots \leq Y_{2n+1}$. (In other words, for each outcome the Y_j are a permutation of the X_i and are in non-decreasing order. These are known as the *order statistics* of the sample.) Calculate expressions for the distribution function and for the probability density function of the random variable Y_{n+1} (the *sample median*).

6. Suppose that n items are being tested simultaneously and that the items hav independent lifetimes, each having the exponential distribution with parameter $\lambda > 0$. Determine the mean and variance of the time until r items have failed.

7. Let X be a real-valued random variable. Suppose that the moment-generating function $m(\theta) = \mathbf{E}(e^{\theta X})$ is finite for some $\theta > 0$. Prove that $\lim_{x\to\infty} x^n \mathbf{P}[X \geq x] = 0$ for every $n \geq 0$.

8. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then the geometric mean lies between the harmonic mean and the arithmetic mean:

$$\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{x_i}\right)^{-1} \leq \left(\prod_{i=1}^n x_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^n x_i.$$

The second inequality is the AM-GM inequality; establish the first one.

9. Find the approximate probability that the number of 6's in 12000 rolls of a fair die is between 1900 and 2150.

10. Find a number c such that the probability is about 1/2 that in 1000 tosses of a fair coin the number of heads lies between 490 and c.

11. Use the central limit theorem to prove that, for any positive real number λ,

$$\frac{\lambda^n}{(n-1)!}\int_0^{\lambda n} x^{n-1}e^{-\lambda x}\,dx \to 0$$

as $n \to \infty$.

12. For Buffon's needle, calculate the probability that the needle intersects a line in the case where the length of the needle is greater than the spacing between the lines.

13. Suppose that X, Y and Z are independent random variables, each uniformly distributed on $(0, 1)$. Prove that $(XY)^Z$ is also uniformly distributed on $(0, 1)$.

14. Suppose that $X_1, X_2, \ldots$ form a sequence of independent random variables, each uniformly distributed on $(0, 1)$. Let $N = \min\{n : X_1 + \ldots + X_n \geq 1\}$. Calculate $\mathbf{P}[N \geq k]$ for each $k \geq 1$ and hence show that $\mathbf{E}N = e$.