Michaelmas Term 2023-24

Numbers and Sets: Examples Sheet 2 of 4

1. Find integers x and y with $76x+45y=1$. Do there exist integers x and y with $3381x+2646y=21$?

2. Let $a,b,c,d \in \mathbb{N}$. Must the numbers $(a,b)(c,d)$ and (ac,bd) be equal? If not, must one be a factor of the other? If $(a,b)=(a,c)=1$, must we have $(a,bc)=1$?

3. Show that a positive integer n is a multiple of 9 if and only if the sum of its digits is a multiple of 9. The number 2^{29} has nine distinct digits. Which digit is missing?

4. The *Fibonacci numbers* $F_1, F_2, F_3, \dots$ are defined by: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$ (so eg. $F_3 = 2$, $F_4 = 3$, $F_5 = 5$). Is F_{2023} even or odd? Is it a multiple of 3?

5. Solve (i.e. find all solutions of) the equations

(i) $7x \equiv 77 \mod 40$;

(ii) $12y \equiv 30 \mod 54$;

(iii) $3z \equiv 2 \mod 17$ and $4z \equiv 3 \mod 19$.

6. An RSA encryption scheme (n,e) has modulus $n=187$ and encoding exponent $e=7$. Find a suitable decoding exponent d. Check your answer (without electronic assistance) by explicitly encoding the number 35 and then decoding the result.

7. By considering the n fractions $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}$ or otherwise, prove that $n = \sum_{d|n} \phi(d)$.

8. Explain (without electronic assistance) why 23 cannot divide $10^{881}-1$.

9. Let p be a prime of the form $3k+2$. Show that if $x^3 \equiv 1 \mod p$, then $x \equiv 1 \mod p$. Deduce that every number is a cube modulo p, that is, $y^3 \equiv a \mod p$ has an integer solution y for all $a \in \mathbb{Z}$.

10. By considering numbers of the form $(2p_1p_2\dots p_k)^2+1$, prove that there are infinitely many primes of the form $4n+1$.

11. What is the 5th-last digit of $5^{5^{5^{5^5}}}$?

12. Is there a positive integer n for which $n^7 - 77$ is a Fibonacci number?

$^+$13. Let p be a prime. Show that for every set of $2p-1$ integers, one can choose a subset of size p whose sum is divisible by p. Is the same true when the prime p is replaced by a composite integer n?

$^+$14. Let n and k be positive integers. Suppose that n is a kth power modulo p for all primes p. Must n be a kth power?

julia.wolf@dpmms.cam.ac.uk 21st October 2023