Numbers and Sets: Examples Sheet 1 of 4

1. The numbers $3, 5, 7$ are all prime. Does it ever happen again that three numbers of the form $n, n+2, n+4$ are all prime?

2. There are four primes between 0 and 10 and between 10 and 20. Does it ever happen again that there are four primes between two consecutive multiples of 10?

3. Consider the sequence $41, 43, 47, 53, 61, \ldots$ (where each difference is 2 more than the previous one). Are all of these numbers prime?

4. Does there exist a block of 100 consecutive positive integers, none of which is prime?

5. Translate the following sentence into a short English one. Is it true? Write down its negation in symbolic form. (Here m, n, a, b should be understood as ranging over all natural numbers.)

$$\forall m \, \exists n \, \forall a \, \forall b \, (n \geq m) \wedge [(a = 1) \vee (b = 1) \vee (ab \neq n)]$$

6. Show that $2^{19} + 5^{40}$ is not prime. Show also that $2^{91} - 1$ is not prime.

7. If n^2 is a multiple of 3, must n be a multiple of 3?

8. Show that for every positive integer n the number $3^{3n+4} + 7^{2n+1}$ is a multiple of 11.

9. Write down carefully (while not looking at your notes) a proof that there are infinitely many primes. By considering numbers of the form $4p_1p_2 \ldots p_k - 1$, prove that there are infinitely many primes of the form $4n - 1$. What would go wrong if we tried a similar proof to show that there are infinitely many primes of the form $4n + 1$?

10. Prove that $2^{2^n} - 1$ has at least n distinct prime factors.

11. We are given an operation $*$ on the positive integers, satisfying

(i) $1 * n = n + 1$ for all n;
(ii) $m * 1 = (m-1) * 2$ for all $m > 1$;
(iii) $m * n = (m-1) * (m * (n-1))$ for all $m, n > 1$.

Find the value of $5 * 5$.

12. Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be?

13. The *repeat* of a positive integer is obtained by writing it twice in a row. For example, the repeat of 254 is 254254. Is there a positive integer whose repeat is a square number?

14. Let $a < b$ be distinct natural numbers. Prove that every block of b consecutive natural numbers contains two distinct numbers whose product is a multiple of ab. $^+$ Suppose now that $a < b < c$. Must every block of c consecutive natural numbers contain three distinct numbers whose product is a multiple of abc?

$^{+}$15. All integers greater than one but less than 100 are put into a hat and two are drawn. Sophie is given their sum and Paul their product. Sophie says, "I can tell you don't know the numbers." Paul replies, "Now I do." Sophie exclaims, "Now I do too!" What are the numbers?