

Numbers and Sets: Examples Sheet 1 of 4

1. The numbers 3, 5, 7 are all prime. Does it ever happen again that three numbers of the form $n, n + 2, n + 4$ are all prime?
2. There are four primes between 0 and 10 and between 10 and 20. Does it ever happen again that there are four primes between two consecutive multiples of 10?
3. Consider the sequence 41, 43, 47, 53, 61, ... (where each difference is 2 more than the previous one). Are all of these numbers prime?
4. Does there exist a block of 100 consecutive positive integers, none of which is prime?
5. Translate the following sentence into a short English one. Is it true? Write down its negation in symbolic form. (Here m, n, a, b should be understood as ranging over all natural numbers.)

$$\forall m \exists n \forall a \forall b (n \geq m) \wedge [(a = 1) \vee (b = 1) \vee (ab \neq n)]$$

6. Show that $2^{19} + 5^{40}$ is not prime. Show also that $2^{91} - 1$ is not prime.
7. If n^2 is a multiple of 3, must n be a multiple of 3?
8. Show that for every positive integer n the number $3^{3n+4} + 7^{2n+1}$ is a multiple of 11.
9. Write down carefully (while not looking at your notes) a proof that there are infinitely many primes. By considering numbers of the form $4p_1p_2 \dots p_k - 1$, prove that there are infinitely many primes of the form $4n - 1$. What would go wrong if we tried a similar proof to show that there are infinitely many primes of the form $4n + 1$?
10. Prove that $2^{2^n} - 1$ has at least n distinct prime factors.
11. We are given an operation $*$ on the positive integers, satisfying
 - (i) $1 * n = n + 1$ for all n ;
 - (ii) $m * 1 = (m - 1) * 2$ for all $m > 1$;
 - (iii) $m * n = (m - 1) * (m * (n - 1))$ for all $m, n > 1$.

Find the value of $5 * 5$.

12. Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be?
13. The *repeat* of a positive integer is obtained by writing it twice in a row. For example, the repeat of 254 is 254254. Is there a positive integer whose repeat is a square number?
14. Let $a < b$ be distinct natural numbers. Prove that every block of b consecutive natural numbers contains two distinct numbers whose product is a multiple of ab .⁺ Suppose now that $a < b < c$. Must every block of c consecutive natural numbers contain three distinct numbers whose product is a multiple of abc ?

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+15. All integers greater than one but less than 100 are put into a hat and two are drawn. Sophie is given their sum and Paul their product. Sophie says, "I can tell you don't know the numbers." Paul replies, "Now I do." Sophie exclaims, "Now I do too!" What are the numbers?