

IA Numbers & Sets – Example Sheet 2

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Questions marked † are more challenging.

- Find integers x and y with $76x + 45y = 1$. Do there exist integers x and y with $3528x + 966y = 24$?
- Let $a, b, c, d \in \mathbb{N}$. Must the numbers $\text{hcf}(a, b) \cdot \text{hcf}(c, d)$ and $\text{hcf}(ac, bd)$ be equal? If not, must one be a factor of the other? If $\text{hcf}(a, b) = \text{hcf}(a, c) = 1$, must we have $\text{hcf}(a, bc) = 1$?
- Write down a mathematical equation, with a few sentences of justification, that explains why your birthday can fall on any day of the calendar week (i.e. Monday to Sunday).
- Show that a positive integer n is a multiple of 9 if and only if the sum of its digits is a multiple of 9. The number 2^{29} has nine distinct digits. Which digit is missing?
- The *Fibonacci numbers* $F_1, F_2, F_3, \dots$ are defined by: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$ (so eg. $F_3 = 2$, $F_4 = 3$, $F_5 = 5$). Is F_{2026} even or odd? Is it a multiple of 3?
Show that any two consecutive Fibonacci numbers are coprime.
- Solve (i.e. find all solutions of) the equations
 - $7x \equiv 77 \pmod{40}$;
 - $12y \equiv 30 \pmod{54}$;
 - $3z \equiv 2 \pmod{17}$ and $4z \equiv 3 \pmod{19}$.
- An RSA encryption scheme (n, e) has modulus $n = 187$ and encoding exponent $e = 7$. Find a suitable decoding exponent d . Check your answer (without electronic assistance) by explicitly encoding the number 35 and then decoding the result.
- By considering the n fractions $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}$ or otherwise, prove that $n = \sum_{d|n} \varphi(d)$.
- Evaluate $20! \cdot 21^{20} \pmod{23}$. Explain why 23 cannot divide $10^{881} - 1$.
- Let p be a prime of the form $3k + 2$. Show that if $x^3 \equiv 1 \pmod{p}$, then $x \equiv 1 \pmod{p}$. Deduce that every number is a cube modulo p , that is, $y^3 \equiv a \pmod{p}$ has an integer solution y for all $a \in \mathbb{Z}$.
- By considering numbers of the form $(2p_1 p_2 \dots p_k)^2 + 1$, prove that there are infinitely many primes of the form $4n + 1$.
- What is the 5th-last digit of $5^{5^{5^5}}$?
- Is there a positive integer n for which $n^7 - 77$ is a Fibonacci number?
- † Let n and k be positive integers. Suppose that n is a k th power modulo p for all primes p . Must n be a k th power?