IA Numbers & Sets – Example Sheet 1

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Questions marked † are more challenging.

1. The numbers $3, 5, 7$ are all prime. Does it ever happen again that three numbers of the form $n, n+2, n+4$ are all prime?

2. There are four primes between 0 and 10 and between 10 and 20. Does it ever happen again that there are four primes between two consecutive multiples of 10?

3. Consider the sequence $41, 43, 47, 53, 61, \ldots$ (where each difference is 2 more than the previous one). Are all of these numbers prime?

4. Translate the following sentence into a short English one. Is it true? Write down its negation in symbolic form. (Here m, n, a, b should be understood as ranging over all natural numbers.)

$$\forall m\ \exists n\ \forall a\ \forall b\ (n \geq m) \wedge [(a = 1) \vee (b = 1) \vee (ab \neq n)]$$

5. Find a prime factor of $2^{19} + 5^{40}$. Can you find another? Show that $2^{91} - 1$ is not prime.

6. If n^2 is a multiple of 3, must n be a multiple of 3?

7. Use induction to show that for every positive integer n the number $2^{n+2} + 3^{2n+1}$ is a multiple of 7.

8. Write down carefully (while not looking at your notes) a proof that there are infinitely many primes. By considering numbers of the form $4p_1p_2 \ldots p_k - 1$, prove that there are infinitely many primes of the form $4n - 1$. What would go wrong if we tried a similar proof to show that there are infinitely many primes of the form $4n + 1$?

9. Prove that $2^{2^n} - 1$ has at least n distinct prime factors.

10. We are given an operation $*$ on the positive integers, satisfying

    (i) $1 * n = n + 1$ for all n;

    (ii) $m * 1 = (m - 1) * 2$ for all $m > 1$;

    (iii) $m * n = (m - 1) * (m * (n - 1))$ for all $m, n > 1$.

    Find the value of $5 * 5$.

11. Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be?

12. Let $a < b$ be distinct natural numbers. Prove that every block of b consecutive natural numbers contains two distinct numbers whose product is a multiple of ab.

    † Show by example that if $a < b < c$ are natural numbers, then a block of c consecutive numbers need not contain three distinct numbers whose product is a multiple of abc.

13. *Fun bonus question:* All integers greater than one but less than 100 are put into a hat and two are drawn. Sabrina is given their sum and Peter their product. Sabrina says, "I can tell you don't know the numbers." Peter replies, "Now I do." Sabrina exclaims, "Now I do too!" What are the numbers?

14. *Fun bonus question:* Each card below has a number on one side and colour on the other. Which card(s) must be turned over to test the idea that if a card shows an even number on one face, then its opposite face is blue?