

Numbers and Sets: Examples Sheet 2 of 4

1. Find integers x and y with $76x + 45y = 1$. Do there exist integers x and y with $3381x + 2646y = 21$?
2. Let $a, b, c, d \in \mathbb{N}$. Must the numbers $(a, b)(c, d)$ and (ac, bd) be equal? If not, must one be a factor of the other? If $(a, b) = (a, c) = 1$, must we have $(a, bc) = 1$?
3. Show that a positive integer n is a multiple of 9 if and only if the sum of its digits is a multiple of 9. The number 2^{29} has nine distinct digits. Which digit is missing?
4. The *Fibonacci numbers* F_1, F_2, F_3, \dots are defined by: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$ (so eg. $F_3 = 2$, $F_4 = 3$, $F_5 = 5$). Is F_{2021} even or odd? Is it a multiple of 3?
5. Solve (i.e. find all solutions of) the equations
 - (i) $7x \equiv 77 \pmod{40}$;
 - (ii) $12y \equiv 30 \pmod{54}$;
 - (iii) $3z \equiv 2 \pmod{17}$ and $4z \equiv 3 \pmod{19}$.
6. An RSA encryption scheme (n, e) has modulus $n = 187$ and coding exponent $e = 7$. Find a suitable decoding exponent d . Check your answer (without electronic assistance) by explicitly encoding the number 35 and then decoding the result.
7. By considering the n fractions $\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}$ or otherwise, prove that $n = \sum_{d|n} \phi(d)$.
8. Explain (without electronic assistance) why 23 cannot divide $10^{881} - 1$.
9. Let p be a prime of the form $3k + 2$. Show that if $x^3 \equiv 1 \pmod{p}$, then $x \equiv 1 \pmod{p}$. Deduce that every number is a cube modulo p , that is, $y^3 \equiv a \pmod{p}$ has an integer solution y for all $a \in \mathbb{Z}$.
10. By considering numbers of the form $(2p_1 p_2 \dots p_k)^2 + 1$, prove that there are infinitely many primes of the form $4n + 1$.
11. What is the 5th-last digit of $5^{5^{5^5}}$?
12. Is there a positive integer n for which $n^7 - 77$ is a Fibonacci number?
- +13. Let p be a prime. Show that for every set of $2p - 1$ integers, one can choose a subset of size p whose sum is divisible by p . Is the same true when the prime p is replaced by a composite integer n ?
- +14. Let n and k be positive integers. Suppose that n is a k th power modulo p for all primes p . Must n be a k th power?