Michaelmas Term 2021-22

Numbers and Sets: Examples Sheet 4 of 4

1. How many subsets of $\{1, 2, 3, 4\}$ have even size? Based on your answer, guess and prove a formula for the number of subsets of $\{1, 2, \ldots, n\}$ of even size.

2. By suitably interpreting each side, establish the identities

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k} = \binom{n+1}{k+1}$$

and

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n},$$

for appropriate ranges of the parameters n and k (which you should specify).

3. The *symmetric difference* of two sets A and B is $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Give a direct proof that the operation $\triangle$ is associative, and then another one based on indicator functions mod 2.

4. Use the inclusion-exclusion principle to determine $\phi(1001)$.

5. Let $A_1, A_2, \ldots$ be sets such that for each n we have $A_1 \cap \ldots \cap A_n \neq \emptyset$. Can we have $A_1 \cap A_2 \cap \ldots = \emptyset$?

6. Does $f \circ g$ injective imply f injective? Does it imply g injective? What happens if we replace 'injective' by 'surjective'?

7. Find an injection from $\mathbb{R}^2$ to $\mathbb{R}$. Is there an injection from the set of all real sequences to $\mathbb{R}$?

8. Define a relation R on $\mathbb{N}$ by setting aRb if a divides b or b divides a. Is R an equivalence relation?

9. Let $r(n)$ denote the number of equivalence relations on a set with n elements. Show that $2^{n-1} \leq r(n) \leq 2^{(n^2-n)/2}$. Can you give a stronger upper bound?

10. Show that there does not exist an uncountable family of pairwise disjoint discs in the plane. What happens if we replace 'discs' by 'circles'?

11. Let F be the collection of all finite subsets of $\mathbb{N}$. Is F countable?

12. A function $f : \mathbb{N} \to \mathbb{N}$ is *increasing* if $f(n+1) \geq f(n)$ for all n and *decreasing* if $f(n+1) \leq f(n)$ for all n. Is the set of increasing functions countable or uncountable? What about the set of decreasing functions?

13. Let S be a collection of subsets of $\mathbb{N}$ such that for every $A, B \in S$ we have $A \subset B$ or $B \subset A$. Can S be uncountable? Is there an uncountable collection T of subsets of $\mathbb{N}$ such that $A \cap B$ is finite for all distinct $A, B \in T$?

14. Find a bijection from the rationals to the non-zero rationals. Is there such a bijection that is order-preserving (i.e. $x < y$ implies $f(x) < f(y)$)?

$^+$15. Let $d \leq n$ be positive integers, with d even. How many subsets of $\{1, 2, \ldots, n\}$ can we find such that any two have symmetric difference of size at most d?

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