Mich. 2020 **NUMBERS AND SETS – EXAMPLES 4** IBL

1. How many subsets of $\{1, 2, 3, 4\}$ have even size? Based on your answer, guess and prove a formula for the number of subsets of $\{1, 2, \ldots, n\}$ of even size.

2. Prove that if p is prime and $1 \leq k \leq p - 1$ then $\binom{p}{k}$ is a multiple of p. Give two proofs: one based on the formula and one based on looking at the k-subsets of $\mathbb{Z}_p$.

3. The *symmetric difference* of sets A and B is $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Give two proofs that the operation $\triangle$ is associative: one directly and one based on indicator functions mod 2.

4. Use the inclusion-exclusion principle to determine $\phi(1001)$.

5. Let $A_1, A_2, \ldots$ be sets such that for each n we have $A_1 \cap \ldots \cap A_n \neq \emptyset$. Can we have $A_1 \cap A_2 \cap \ldots = \emptyset$?

6. Does $f \circ g$ injective imply f injective? Does it imply g injective? What happens if we replace 'injective' with 'surjective'?

7. Find an injection from $\mathbb{R}^2$ to $\mathbb{R}$. Is there an injection from the set of all real sequences to $\mathbb{R}$?

8. Define a relation R on $\mathbb{N}$ by setting aRb if a divides b or b divides a. Is R an equivalence relation?

9. Show that there does not exist an uncountable family of pairwise disjoint discs in the plane. What happens if we replace 'discs' by 'circles'?

10. Show that the collection of all finite subsets of $\mathbb{N}$ is countable. What goes wrong if we try to use the diagonal argument to show that it is uncountable?

11. A function $f : \mathbb{N} \to \mathbb{N}$ is *increasing* if $f(n + 1) \geq f(n)$ for all n and *decreasing* if $f(n + 1) \leq f(n)$ for all n. Is the set of increasing functions countable or uncountable? What about the set of decreasing functions?

12. Let S be a collection of subsets of $\mathbb{N}$ such that for every $A, B \in S$ we have $A \subset B$ or $B \subset A$. Can S be uncountable?

13. Find a bijection from the rationals to the non-zero rationals. Is there such a bijection that is order-preserving (ie. $x < y$ implies $f(x) < f(y)$)?

14. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value on every interval – in other words, for every $a < b$ and every c there is an x with $a < x < b$ such that $f(x) = c$.

$^{+}$15. Let $d \leq n$ be positive integers, with d even. How many subsets of $\{1, 2, \ldots, n\}$ can we find such that any two have symmetric difference of size at most d?