Mich. 2019 **NUMBERS AND SETS – EXAMPLES 1** IBL

1. The numbers 3,5,7 are all prime; does it ever happen again that three numbers of the form $n, n+2, n+4$ are all prime?

2. Between 10 and 20 there are 4 primes; does it ever happen again that there are 4 primes between two consecutive multiples of 10 (apart from between 0 and 10)?

3. Consider the sequence $41, 43, 47, 53, 61, \ldots$ (where each difference is 2 more than the previous one). Are all of these numbers prime?

4. Does there exist a block of 100 consecutive positive integers, none of which is prime?

5. Show that $2^{19} + 5^{40}$ is not prime. Show also that $2^{91} - 1$ is not prime.

6. If n^2 is a multiple of 3, must n be a multiple of 3?

7. Show that, for every positive integer n, the number $3^{3n+4} + 7^{2n+1}$ is a multiple of 11.

8. Write down carefully (while not looking at your notes) a proof that there are infinitely many primes. By considering numbers of the form $4p_1p_2 \ldots p_k - 1$, prove that there are infinitely many primes of the form $4n-1$. What would go wrong if we tried a similar proof to show that there are infinitely many primes of the form $4n+1$?

9. Prove that $2^{2^n} - 1$ has at least n distinct prime factors.

10. We are given an operation $*$ on the positive integers, satisfying
(i) $1 * n = n + 1$ for all n
(ii) $m * 1 = (m-1) * 2$ for all $m > 1$
(iii) $m * n = (m-1) * (m * (n-1))$ for all $m, n > 1$.
Find the value of $5 * 5$.

11. Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be?

12. Find a positive integer a such that, for every positive integer n, the number $n^4 + a$ is not prime.

13. The *repeat* of a positive integer is obtained by writing it twice in a row (so for example the repeat of 254 is 254254). Is there a positive integer whose repeat is a square number?

14. Some red sweets and blue sweets are distributed among 99 bags. Gareth wants to select 50 of the bags in such a way that he obtains at least half of the red sweets and at least half of the blue sweets. Is he always able to do this?

$^+$15. Among a group of n dons, any two have exactly one mutual friend. Show that some don is friends with all the others.