Numbers and Sets (2018–19) Example Sheet 4 of 4

1. Define a sequence $(x_n)_{n=1}^{\infty}$ by setting $x_1 = 1$ and $x_{n+1} = \frac{x_n}{1+\sqrt{x_n}}$ for all $n \geq 1$. Show that $(x_n)_{n=1}^{\infty}$ converges, and determine its limit.

2. Let $(a_n)_{n=1}^{\infty}$ be a sequence of reals. Show that if $(a_n)_{n=1}^{\infty}$ is convergent then we must have $a_n - a_{n-1} \to 0$. If $a_n - a_{n-1} \to 0$, must $(a_n)_{n=1}^{\infty}$ be convergent?

3. Let $[a_n, b_n]$, $n = 1, 2, \ldots$, be closed intervals with $[a_n, b_n] \cap [a_m, b_m] \neq \emptyset$ for all n, m. Prove that $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$.

4. Which of the following sequences $(x_n)_{n=1}^{\infty}$ converge?

$$x_n = \frac{3n}{n+3} \qquad x_n = \frac{n^{100}}{2^n} \qquad x_n = \sqrt{n+1} - \sqrt{n} \qquad x_n = (n!)^{1/n}$$

5. Which of the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} \qquad \sum_{n=1}^{\infty} \frac{n!}{n^n} \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}} \qquad \sum_{n=1}^{\infty}{}^{*} \frac{1}{n}$$

In the last case, the $*$ means omit all values of n which, when written in base 10, have some digit equal to 7.

6. Let $a_n \in \mathbb{R}$ and let $b_n = \frac{1}{n}\sum_{i=1}^{n} a_i$. Show that, if $a_n \to a$ as $n \to \infty$, then $b_n \to a$ also.

7. Let $\sum_{n=1}^{\infty} x_n$ be a divergent series, where $x_n > 0$ for all n. Show that there is a divergent series $\sum_{n=1}^{\infty} y_n$ with $y_n > 0$ for all n, such that $y_n / x_n \to 0$.

8. Let $\sum_{n=1}^{\infty} x_n$ be convergent. If $x_n > 0$ for all n, show that $\sum_{n=1}^{\infty} x_n^2$ also converges. What if sometimes $x_n < 0$? What are the corresponding answers for $\sum_{n=1}^{\infty} x_n^3$?

9. Show that any collection of pairwise disjoint discs in the plane is countable. What happens if we replace 'discs' by 'circles'?

10. Let $\mathcal{F}$ be the set of all finite subsets of $\mathbb{N}$. Is $\mathcal{F}$ countable?

11. A function $f : \mathbb{N} \to \mathbb{N}$ is *increasing* if $f(n+1) \geq f(n)$ for all n and *decreasing* if $f(n+1) \leq f(n)$ for all n. Is the set of increasing functions countable or uncountable? What about the set of decreasing functions?

12. Find an injection $\mathbb{R}^2 \to \mathbb{R}$. Is there an injection from the set of all real sequences to $\mathbb{R}$?

13. Let $S \subset \mathcal{P}\mathbb{N}$ be such that if $A, B \in S$ then $A \subset B$ or $B \subset A$. Can S be uncountable? Is there an uncountable family $T \subset \mathcal{P}\mathbb{N}$ such that $A \cap B$ is finite for all distinct $A, B \in T$?

14. For each $x \in \mathbb{R}$ we are given an interval $I_x = [x - \delta_x, x + \delta_x]$ with $\delta_x \geq 0$. Moreover, for each $x, y \in \mathbb{R}$ with $y \in I_x$, we have $\delta_y < \delta_x$. Show that $\delta_x = 0$ for uncountably many x.

* **15.** Let $(x_n)_{n=1}^{\infty}$ be a real sequence with $x_n \to 0$. Prove carefully that we may choose $(\epsilon_n)_{n=1}^{\infty}$, with each $\epsilon_n = \pm 1$, such that $\sum_{n=1}^{\infty} \epsilon_n x_n$ is convergent. If $(y_n)_{n=1}^{\infty}$ is another real sequence tending to 0, can we choose the ϵ_n so that $\sum_{n=1}^{\infty} \epsilon_n y_n$ is convergent as well?

* **16.** Is there an enumeration of $\mathbb{Q}$ as $q_1, q_2, q_3, \ldots$ such that $\sum (q_n - q_{n+1})^2$ converges?